Fundamentals of Thermal-Fluid Sciences, 3rd Edition
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SECOND UNIT

TRANSIENT HEAT CONDUCTION

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Objectives

- Assess when the spatial variation of temperature is negligible, and temperature varies nearly uniformly with time, making the simplified lumped system analysis applicable
- Obtain analytical solutions for transient one-dimensional conduction problems in rectangular, cylindrical, and spherical geometries using the method of separation of variables, and understand why a one-term solution is usually a reasonable approximation
- Solve the transient conduction problem in large mediums using the similarity variable, and predict the variation of temperature with time and distance from the exposed surface
- Construct solutions for multi-dimensional transient conduction problems using the product solution approach.

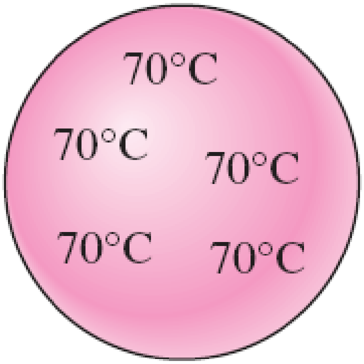
LUMPED SYSTEM ANALYSIS

Interior temperature of some bodies remains essentially uniform at all times during a heat transfer process.

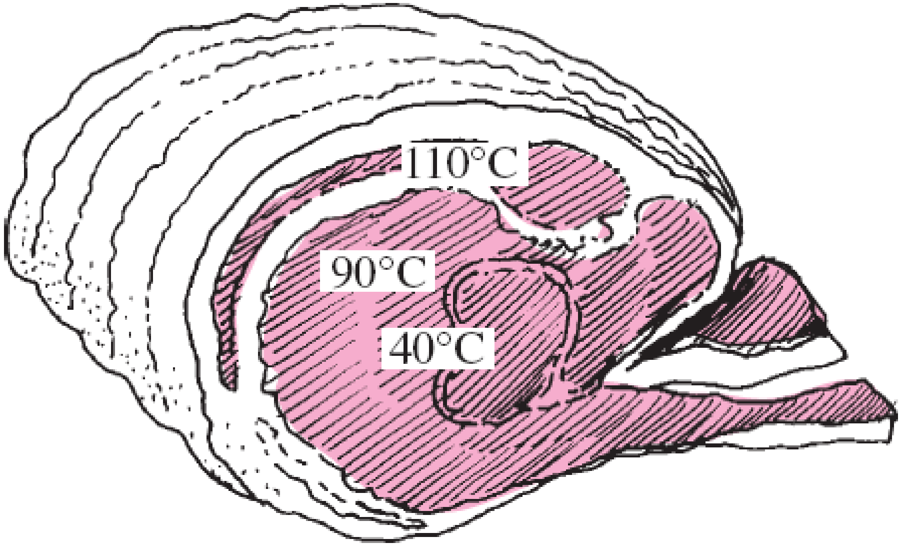
The temperature of such bodies can be taken to be a function of time only, $T(t)$.

Heat transfer analysis that utilizes this idealization is known as **lumped system analysis**.

A small copper ball can be modeled as a lumped system, but a roast beef cannot.



(a) Copper ball



(b) Roast beef

$$\begin{pmatrix}\text{Heat transfer into the body} \\ \text{during } dt\end{pmatrix} = \begin{pmatrix}\text{The increase in the} \\ \text{energy of the body} \\ \text{during } dt\end{pmatrix}$$

$$hA_s(T_\infty - T)\,dt = mc_p\,dT$$

$$m = \rho V \qquad dT = d(T - T_\infty)$$

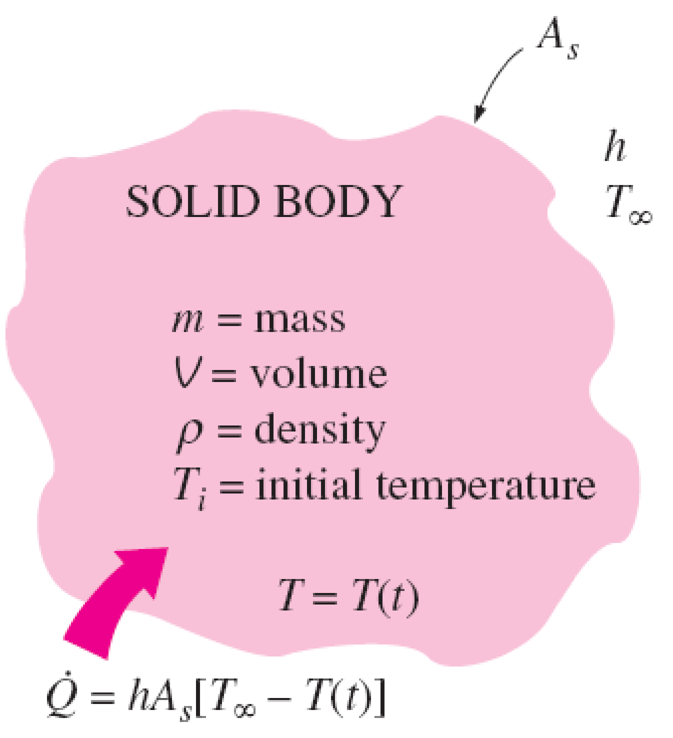
$$\frac{d(T - T_\infty)}{T - T_\infty} = -\frac{hA_s}{\rho V c_p}\,dt$$

Integrating with
$T = T_i$ at $t = 0$
$T = T(t)$ at $t = t$

$$\ln\frac{T(t) - T_\infty}{T_i - T_\infty} = -\frac{hA_s}{\rho V c_p}\,t$$

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt} \qquad b = \frac{hA_s}{\rho V c_p} \quad (1/\text{s})$$

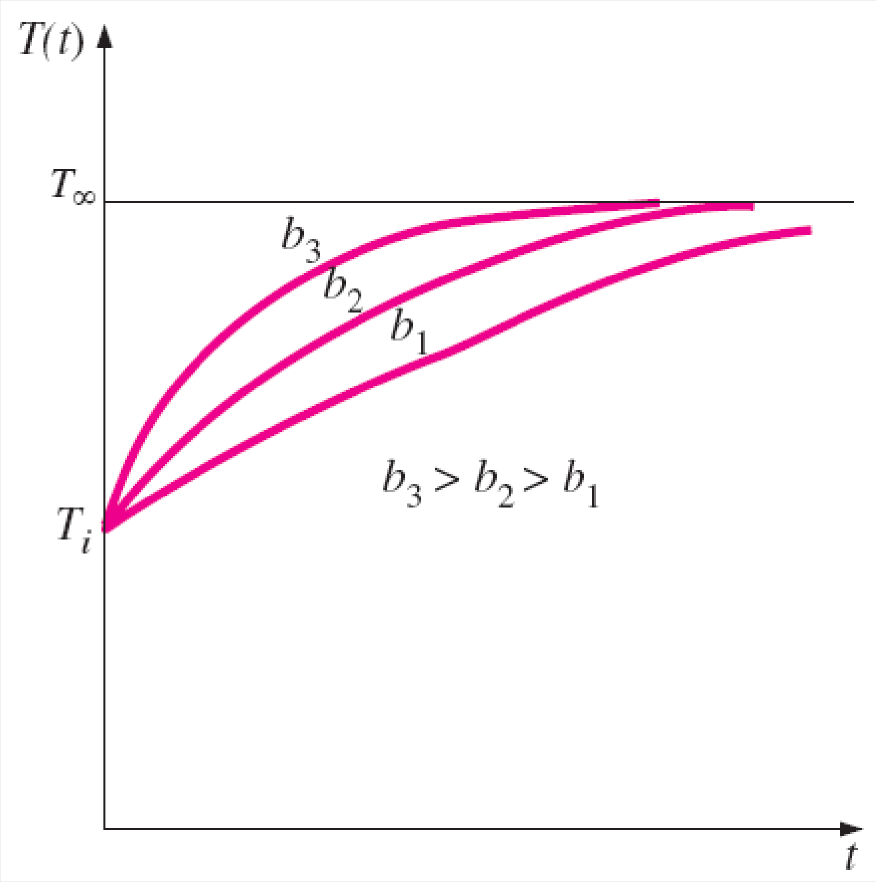
time constant



The geometry and parameters involved in the lumped system analysis.

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt}$$

$$b = \frac{hA_s}{\rho V c_p}$$



The temperature of a lumped system approaches the environment temperature as time gets larger.

- This equation enables us to determine the temperature $T(t)$ of a body at time t, or alternatively, the time t required for the temperature to reach a specified value $T(t)$.
- The temperature of a body approaches the ambient temperature T_∞ exponentially.
- The temperature of the body changes rapidly at the beginning, but rather slowly later on. A large value of b indicates that the body approaches the environment temperature in a short time

$$\dot{Q}(t) = hA_s[T(t) - T_\infty] \qquad \text{(W)}$$

The *rate* of convection heat transfer between the body and its environment at time *t*

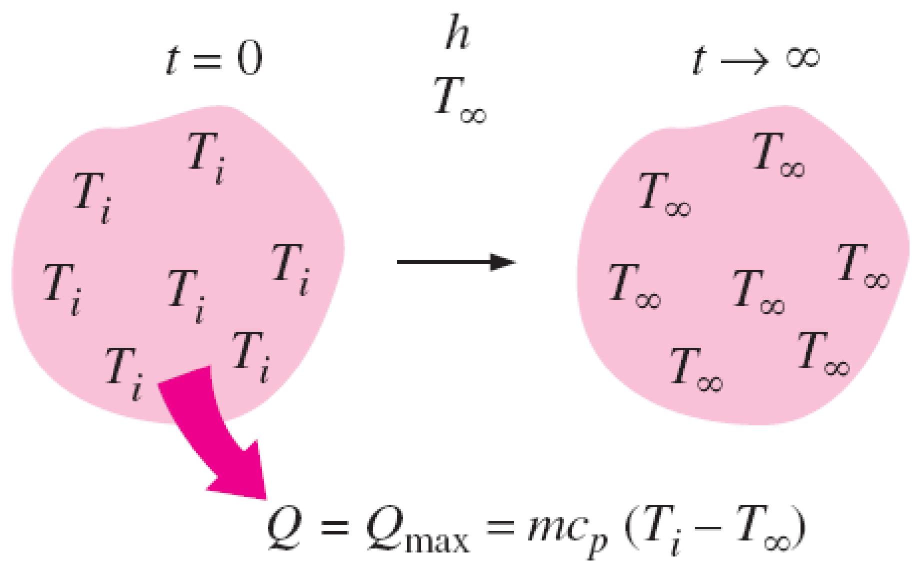
$$Q = mc_p[T(t) - T_i] \qquad \text{(kJ)}$$

The *total amount* of heat transfer between the body and the surrounding medium over the time interval $t = 0$ to t

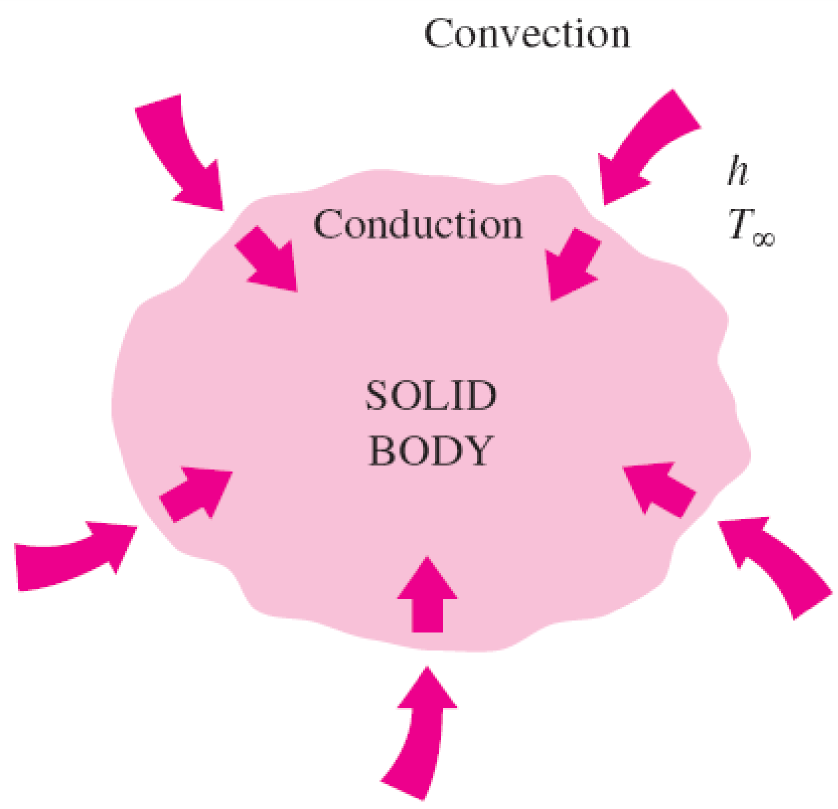
$$Q_{\text{max}} = mc_p(T_\infty - T_i) \qquad \text{(kJ)}$$

The *maximum* heat transfer between the body and its surroundings

Heat transfer to or from a body reaches its maximum value when the body reaches the environment temperature.



Criteria for Lumped System Analysis



$$\text{Bi} = \frac{\text{heat convection}}{\text{heat conduction}}$$

$$L_c = \frac{V}{A_s}$$ Characteristic length

$$\text{Bi} = \frac{hL_c}{k}$$ Biot number

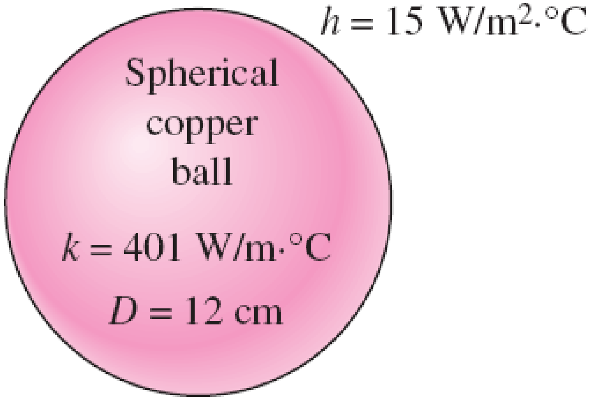
Lumped system analysis is *applicable* if

$$\text{Bi} \leq 0.1$$

When Bi ≤ 0.1, the temperatures within the body relative to the surroundings (i.e., $T - T_\infty$) remain within 5 percent of each other.

$$\text{Bi} = \frac{h}{k/L_c}\frac{\Delta T}{\Delta T} = \frac{\text{Convection at the surface of the body}}{\text{Conduction within the body}}$$

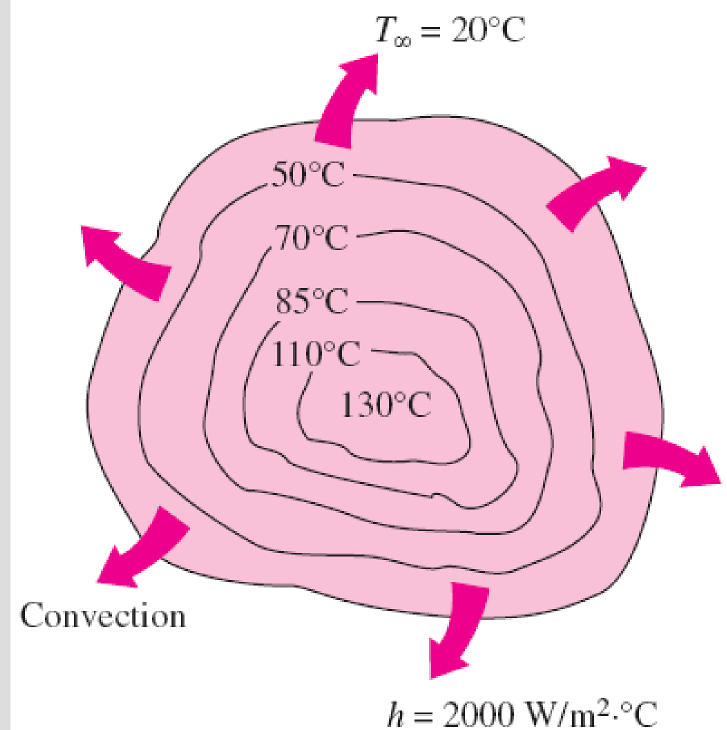
$$\text{Bi} = \frac{L_c/k}{1/h} = \frac{\text{Conduction resistance within the body}}{\text{Convection resistance at the surface of the body}}$$



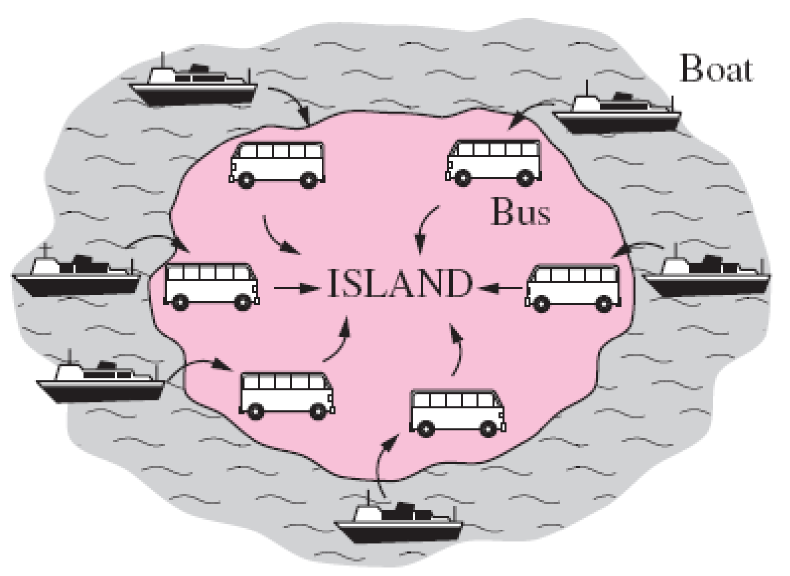
$$L_c = \frac{V}{A_s} = \frac{\frac{1}{6}\pi D^3}{\pi D^2} = \frac{1}{6}D = 0.02 \text{ m}$$

$$\text{Bi} = \frac{hL_c}{k} = \frac{15 \times 0.02}{401} = 0.00075 < 0.1$$

Small bodies with high thermal conductivities and low convection coefficients are most likely to satisfy the criterion for lumped system analysis.



When the convection coefficient *h* is high and *k* is low, large temperature differences occur between the inner and outer regions of a large solid.



Analogy between heat transfer to a solid and passenger traffic to an island.

Example:

The temperature of a gas stream is to be measured by a thermocouple whose junction can be approximated as a 1-mm-diameter sphere, as shown in Fig. 4–9. The properties of the junction are $k = 35$ W/m · °C, $\rho = 8500$ kg/m^3, and $C_p = 320$ J/kg · °C, and the convection heat transfer coefficient between the junction and the gas is $h = 210$ W/m^2 · °C. Determine how long it will take for the thermocouple to read 99 percent of the initial temperature difference.

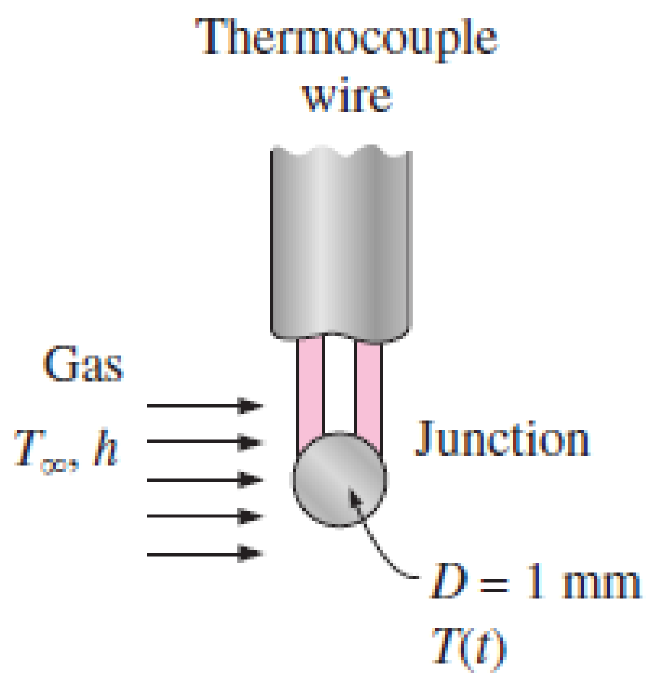
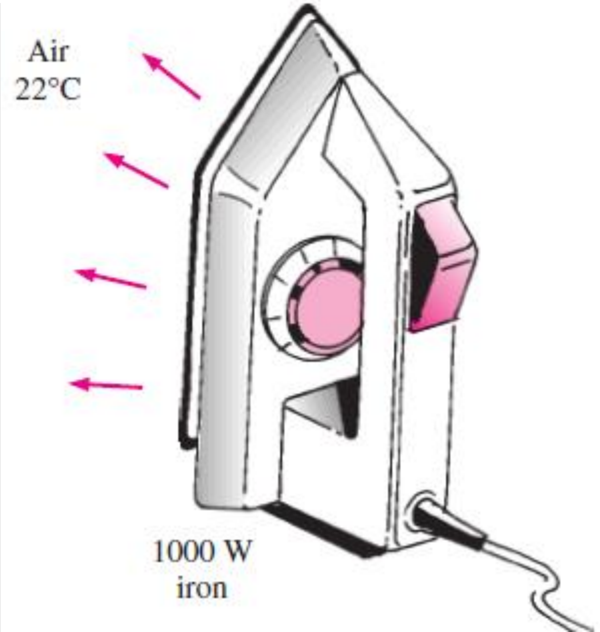


FIGURE 4–9

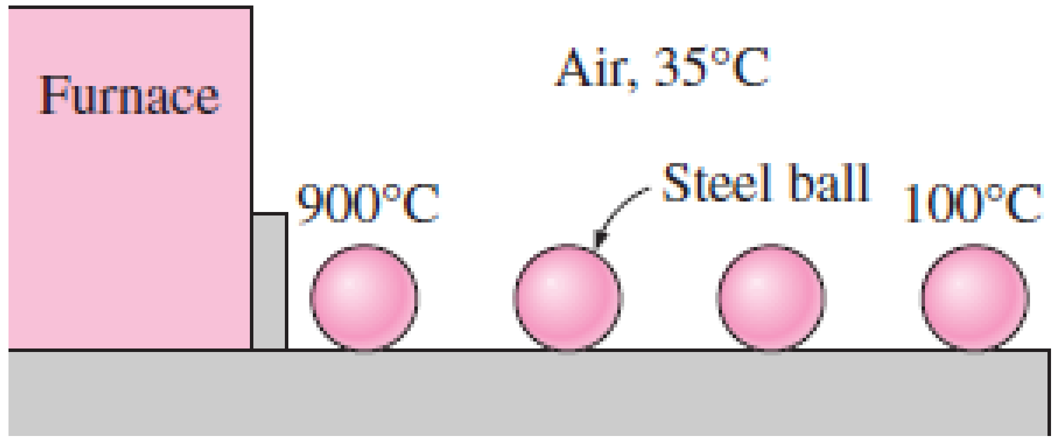
Example:

4–20 Consider a 1000-W iron whose base plate is made of 0.5-cm-thick aluminum alloy 2024-T6 ($\rho = 2770$ kg/m^3, $C_p = 875$ J/kg · °C, $\alpha = 7.3 \times 10^{-5}$ m^2/s). The base plate has a surface area of 0.03 m^2. Initially, the iron is in thermal equilibrium with the ambient air at 22°C. Taking the heat transfer coefficient at the surface of the base plate to be 12 W/m^2 · °C and assuming 85 percent of the heat generated in the resistance wires is transferred to the plate, determine how long it will take for the plate temperature to reach 140°C. Is it realistic to assume the plate temperature to be uniform at all times?



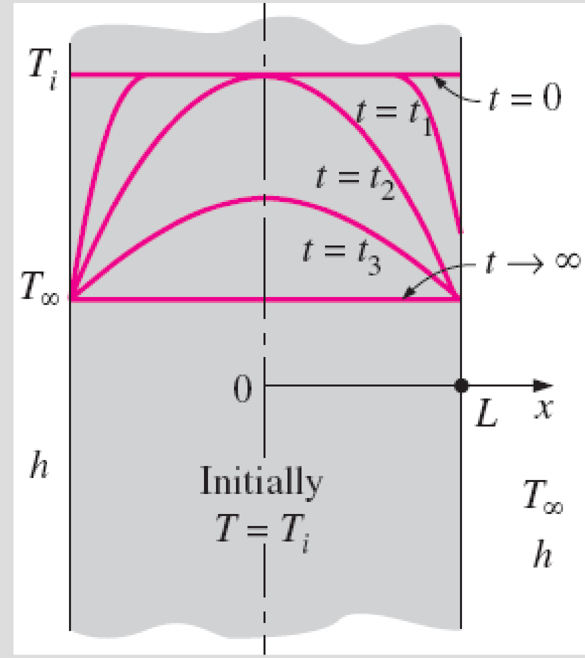
Quiz:

Carbon steel balls (ρ = 7833 kg/m^3, k = 54 W/m · °C, C_p = 0.465 kJ/kg · °C, and α = 1.474 × 10^{-6} m^2/s) 8 mm in diameter are annealed by heating them first to 900°C in a furnace and then allowing them to cool slowly to 100°C in ambient air at 35°C. If the average heat transfer coefficient is 75 W/m^2 · °C, determine how long the annealing process will take. If 2500 balls are to be annealed per hour, determine the total rate of heat transfer from the balls to the ambient air.

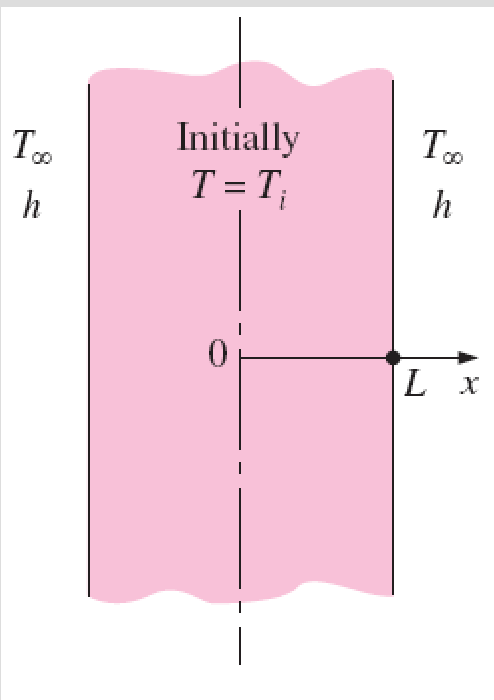


TRANSIENT HEAT CONDUCTION IN LARGE PLANE WALLS, LONG CYLINDERS, AND SPHERES WITH SPATIAL EFFECTS

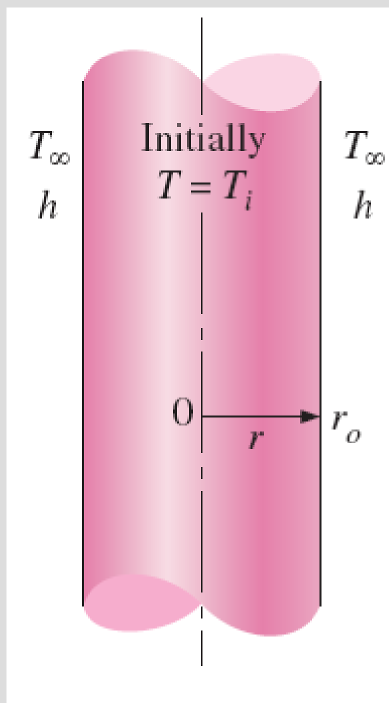
We will consider the variation of temperature with ***time*** and ***position*** in one-dimensional problems such as those associated with a large plane wall, a long cylinder, and a sphere.



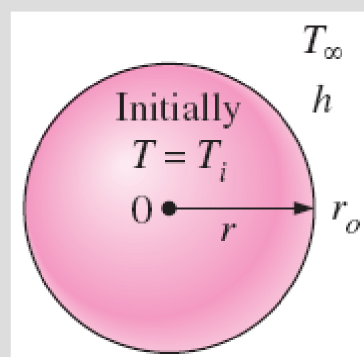
Transient temperature profiles in a plane wall exposed to convection from its surfaces for $T_i > T_\infty$.



(*a*) A large plane wall

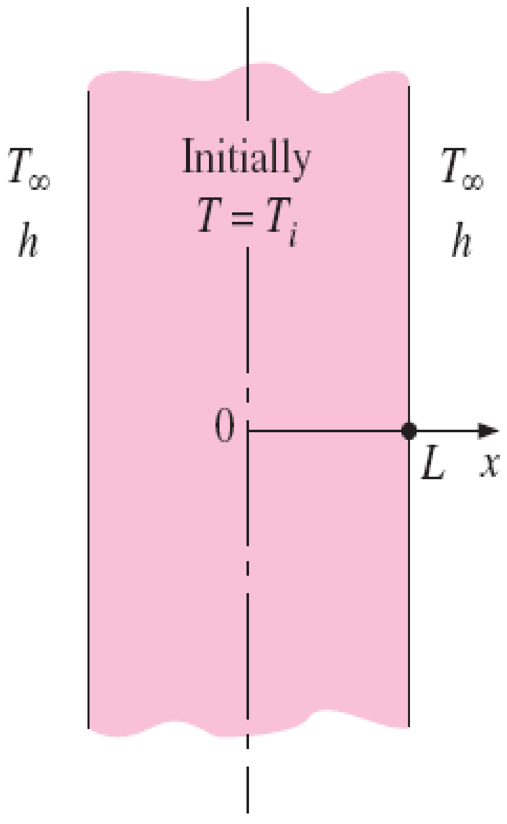


(*b*) A long cylinder



Schematic of the simple geometries in which heat transfer is one-dimensional.

Nondimensionalized One-Dimensional Transient Conduction Problem



(*a*) A large plane wall

Differential equation: $$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Boundary conditions:

$$\frac{\partial T(0, t)}{\partial x} = 0 \quad \text{and} \quad -k\frac{\partial T(L, t)}{\partial x} = h[T(L, t) - T_\infty]$$

Initial condition: $T(x, 0) = T_i$

$\alpha = k/\rho c_p$

$X = x/L$

$\theta(x, t) = [T(x, t) - T_\infty]/[T_i - T_\infty]$

$$\frac{\partial^2 \theta}{\partial X^2} = \frac{L^2}{\alpha}\frac{\partial \theta}{\partial t} \quad \text{and} \quad \frac{\partial \theta(1, t)}{\partial X} = \frac{hL}{k}\theta(1, t)$$

$$\theta(X, \tau) = \frac{T(x, t) - T_i}{T_\infty - T_i}$$ *Dimensionless temperature*

$$X = \frac{x}{L}$$ *Dimensionless distance from the center*

$$\text{Bi} = \frac{hL}{k}$$ *Dimensionless heat transfer coefficient* (Biot number)

$$\tau = \frac{\alpha t}{L^2} = \text{Fo}$$ *Dimensionless time* (Fourier number)

(*a*) Original heat conduction problem:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \quad T(x, 0) = T_i$$

$$\frac{\partial T(0, t)}{\partial x} = 0, \quad -k\frac{\partial T(L, t)}{\partial x} = h[T(L, t) - T_\infty]$$

$$T = F(x, L, t, k, \alpha, h, T_i)$$

(*b*) Nondimensionalized problem:

$$\frac{\partial^2 \theta}{\partial X^2} = \frac{\partial \theta}{\partial \tau}, \quad \theta(X, 0) = 1$$

$$\frac{\partial \theta(0, \tau)}{\partial X} = 0, \quad \frac{\partial \theta(1, \tau)}{\partial X} = -\text{Bi}\,\theta(1, \tau)$$

$$\theta = f(X, \text{Bi}, \tau)$$

Nondimensionalization reduces the number of independent variables in one-dimensional transient conduction problems from 8 to 3, offering great convenience in the presentation of results.

Exact Solution of One-Dimensional Transient Conduction Problem

TABLE 18–1

Summary of the solutions for one-dimensional transient conduction in a plane wall of thickness $2L$, a cylinder of radius r_o and a sphere of radius r_o subjected to convention from all surfaces.*

Geometry	Solution	λ_n's are the roots of
Plane wall	$\theta = \sum_{n=1}^{\infty} \frac{4 \sin \lambda_n}{2\lambda_n + \sin(2\lambda_n)} e^{-\lambda_n^2 \tau} \cos(\lambda_n x / L)$	$\lambda_n \tan \lambda_n = \text{Bi}$
Cylinder	$\theta = \sum_{n=1}^{\infty} \frac{2}{\lambda_n} \frac{J_1(\lambda_n)}{J_0^2(\lambda_n) + J_1^2(\lambda_n)} e^{-\lambda_n^2 \tau} J_0(\lambda_n r / r_o)$	$\lambda_n \frac{J_1(\lambda_n)}{J_0(\lambda_n)} = \text{Bi}$
Sphere	$\theta = \sum_{n=1}^{\infty} \frac{4(\sin \lambda_n - \lambda_n \cos \lambda_n)}{2\lambda_n - \sin(2\lambda_n)} e^{-\lambda_n^2 \tau} \frac{\sin(\lambda_n x / L)}{\lambda_n x / L}$	$1 - \lambda_n \cot \lambda_n = \text{Bi}$

*Here $\theta = (T - T_i)/(T_\infty - T_i)$ is the dimensionless temperature, $\text{Bi} = hL/k$ or hr_o/k is the Biot number, $\text{Fo} = \tau = \alpha t / L^2$ or $\alpha \tau / r_o^2$ is the Fourier number, and J_0 and J_1 are the Bessel functions of the first kind whose values are given in Table 18–3.

$$\theta_n = A_n e^{-\lambda_n^2 \tau} \cos(\lambda_n X)$$

$$A_n = \frac{4 \sin \lambda_n}{2\lambda_n + \sin(2\lambda_n)}$$

$$\lambda_n \tan \lambda_n = \text{Bi}$$

For Bi = 5, $X = 1$, and $t = 0.2$:

n	λ_n	A_n	θ_n
1	1.3138	1.2402	0.22321
2	4.0336	−0.3442	0.00835
3	6.9096	0.1588	0.00001
4	9.8928	−0.876	0.00000

The analytical solutions of transient conduction problems typically involve infinite series, and thus the evaluation of an infinite number of terms to determine the temperature at a specified location and time.

The term in the series solution of transient conduction problems decline rapidly as n and thus λ_n increases because of the exponential decay function with the exponent $-\lambda_n \tau$.

Approximate Analytical and Graphical Solutions

The terms in the series solutions converge rapidly with increasing time, and for $\tau > 0.2$, keeping the first term and neglecting all the remaining terms in the series results in an error under 2 percent.

Solution with *one-term approximation*

Plane wall: $$\theta_{\text{wall}} = \frac{T(x, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \cos(\lambda_1 x/L), \quad \tau > 0.2$$

Cylinder: $$\theta_{\text{cyl}} = \frac{T(r, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} J_0(\lambda_1 r/r_o), \quad \tau > 0.2$$

Sphere: $$\theta_{\text{sph}} = \frac{T(r, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \frac{\sin(\lambda_1 r/r_o)}{\lambda_1 r/r_o}, \quad \tau > 0.2$$

Center of plane wall ($x = 0$): $$\theta_{0,\text{ wall}} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

Center of cylinder ($r = 0$): $$\theta_{0,\text{ cyl}} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

Center of sphere ($r = 0$): $$\theta_{0,\text{ sph}} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

Note that the case $1/\text{Bi} = k/hL = 0$ corresponds to $h \to \infty$, which corresponds to the case of *specified surface temperature* T_∞. That is, the case in which the surfaces of the body are suddenly brought to the temperature T_∞ at $t = 0$ and kept at T_∞ at all times can be handled by setting h to infinity

The temperature of the body changes from the initial temperature T_i to the temperature of the surroundings T_∞ at the end of the transient heat conduction process. Thus, the *maximum* amount of heat that a body can gain (or lose if $T_i > T_\infty$) is simply the *change* in the *energy content* of the body. That is,

$$Q_{\max} = mC_p(T_\infty - T_i) = \rho V C_p(T_\infty - T_i)$$

TABLE 18–2

Coefficients used in the one-term approximate solution of transient one-dimensional heat conduction in plane walls, cylinders, and spheres (Bi = hL/k for a plane wall of thickness $2L$, and Bi = hr_o/k for a cylinder or sphere of radius r_o)

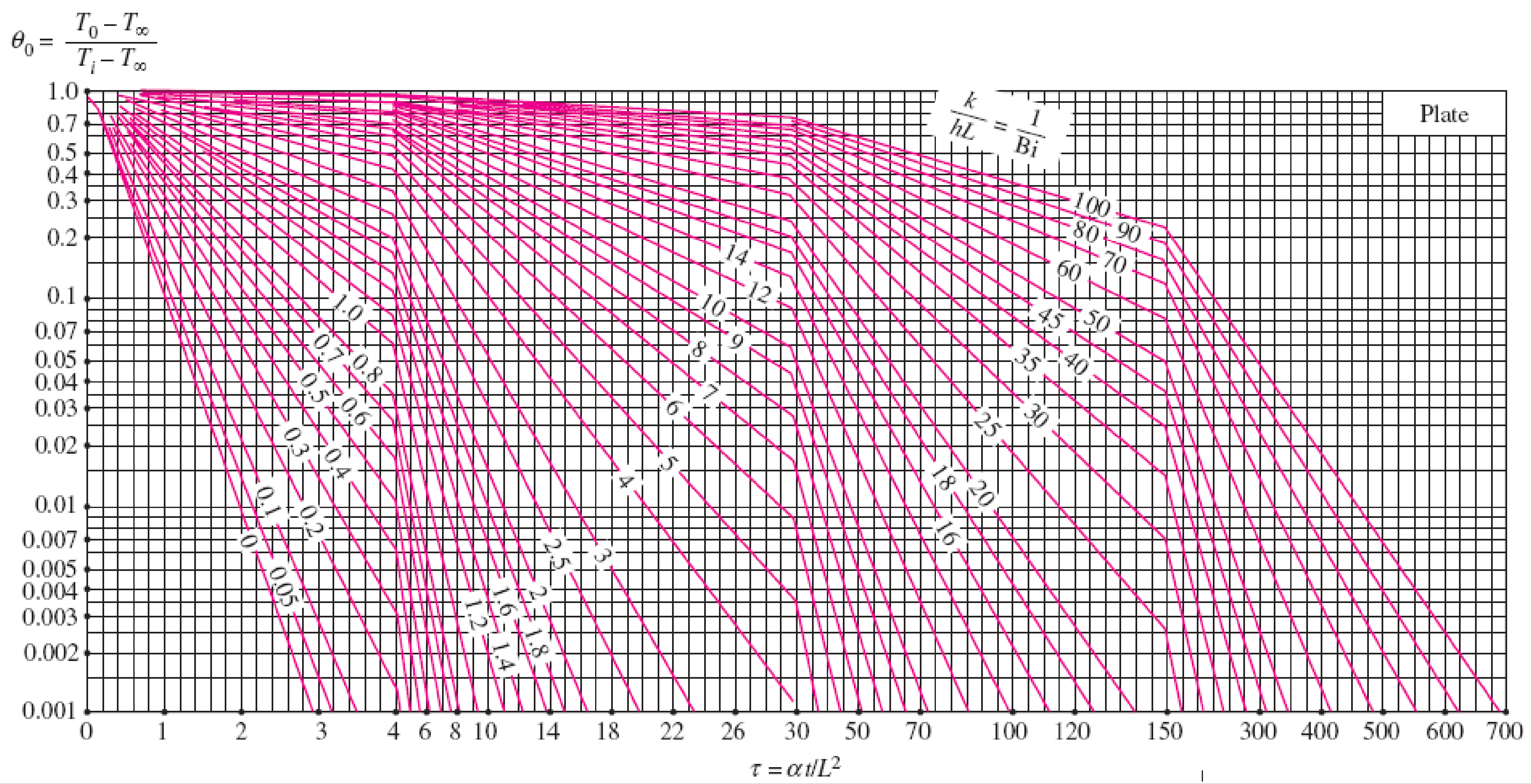
	Plane Wall		Cylinder		Sphere	
Bi	λ_1	A_1	λ_1	A_1	λ_1	A_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.1	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.2	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.3	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0931	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0184	1.1345	1.2644	1.1713
0.7	0.7506	1.0918	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1785	1.5995	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5.0	1.3138	1.2403	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8.0	1.3978	1.2570	2.1286	1.5526	2.7654	1.8920
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2880	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5708	1.2732	2.4048	1.6021	3.1416	2.0000

TABLE 18–3

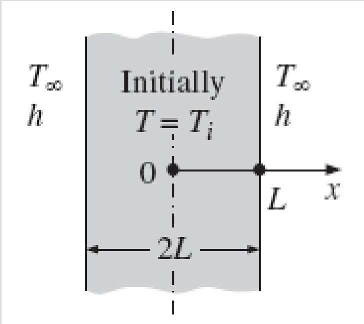
The zeroth- and first-order Bessel functions of the first kind

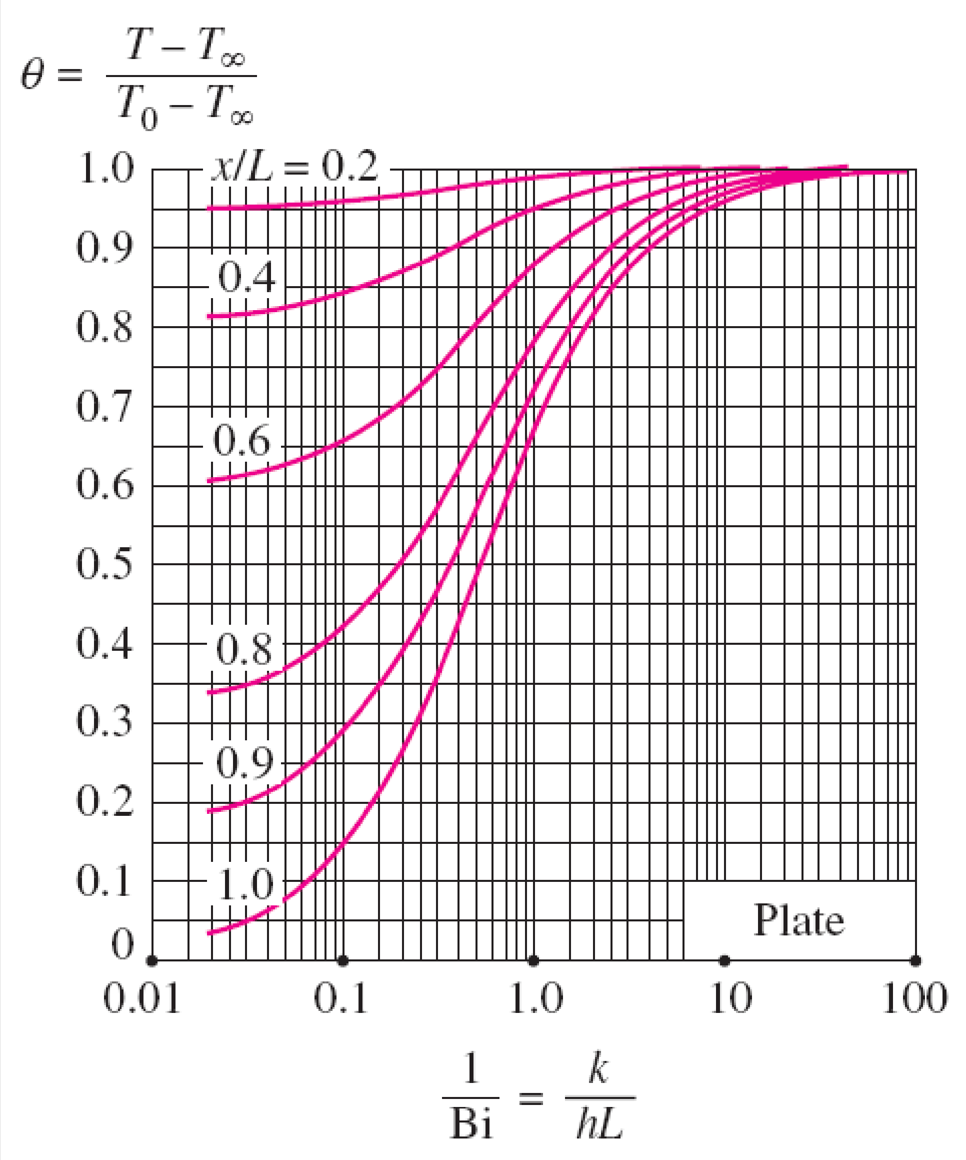
η	$J_0(\eta)$	$J_1(\eta)$
0.0	1.0000	0.0000
0.1	0.9975	0.0499
0.2	0.9900	0.0995
0.3	0.9776	0.1483
0.4	0.9604	0.1960
0.5	0.9385	0.2423
0.6	0.9120	0.2867
0.7	0.8812	0.3290
0.8	0.8463	0.3688
0.9	0.8075	0.4059
1.0	0.7652	0.4400
1.1	0.7196	0.4709
1.2	0.6711	0.4983
1.3	0.6201	0.5220
1.4	0.5669	0.5419
1.5	0.5118	0.5579
1.6	0.4554	0.5699
1.7	0.3980	0.5778
1.8	0.3400	0.5815
1.9	0.2818	0.5812
2.0	0.2239	0.5767
2.1	0.1666	0.5683
2.2	0.1104	0.5560
2.3	0.0555	0.5399
2.4	0.0025	0.5202
2.6	−0.0968	−0.4708
2.8	−0.1850	−0.4097
3.0	−0.2601	−0.3391
3.2	−0.3202	−0.2613

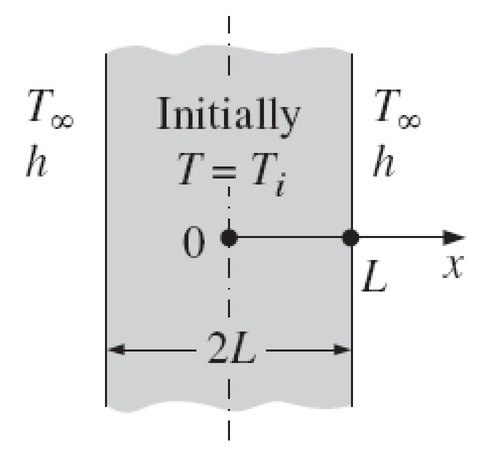
(*a*) Midplane temperature



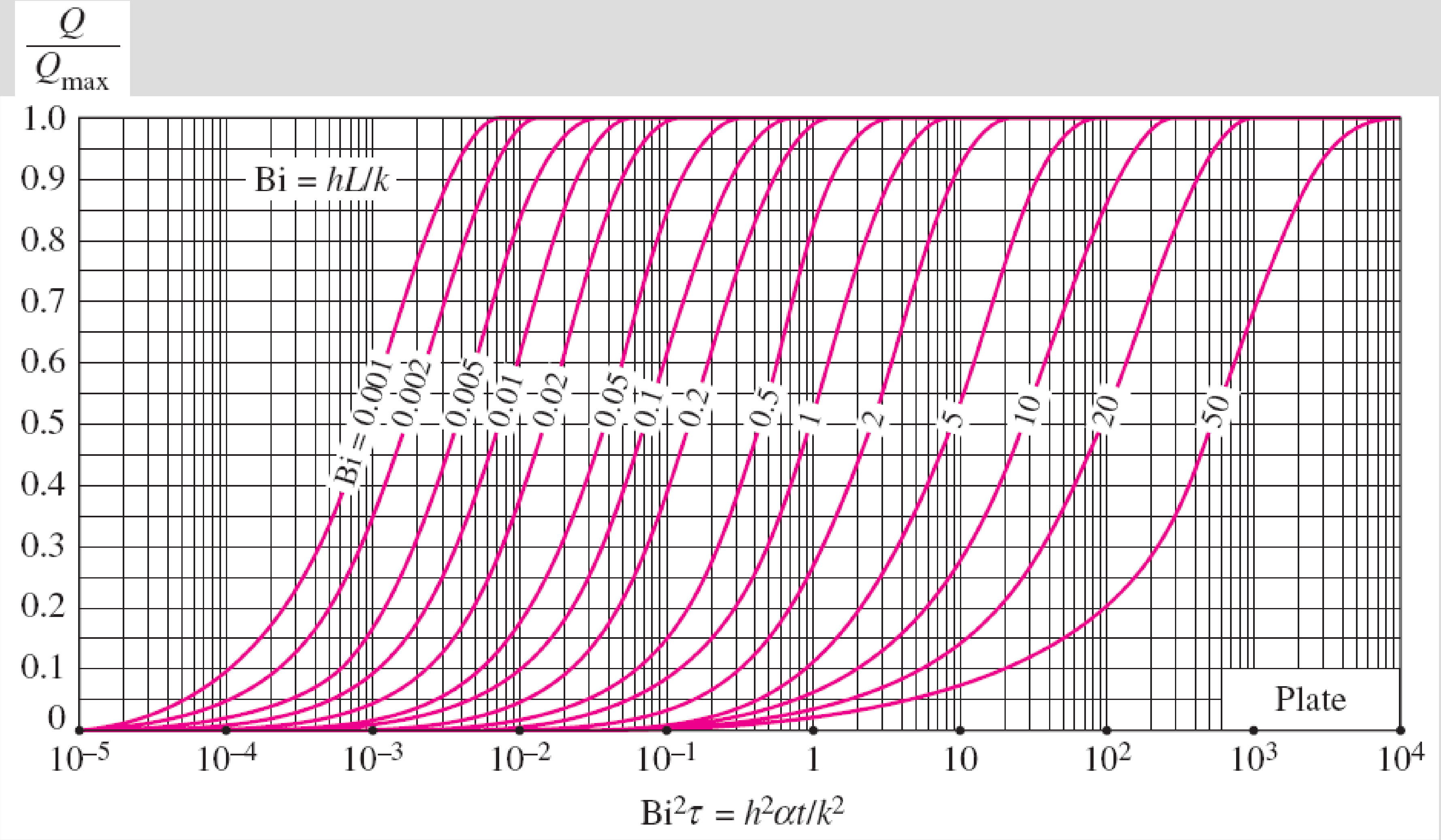
Transient temperature and heat transfer charts (Heisler and Grober charts) for a plane wall of thickness 2*L* initially at a uniform temperature T_i subjected to convection from both sides to an environment at temperature T_∞ with a convection coefficient of *h.*



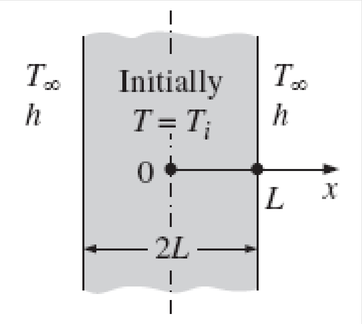


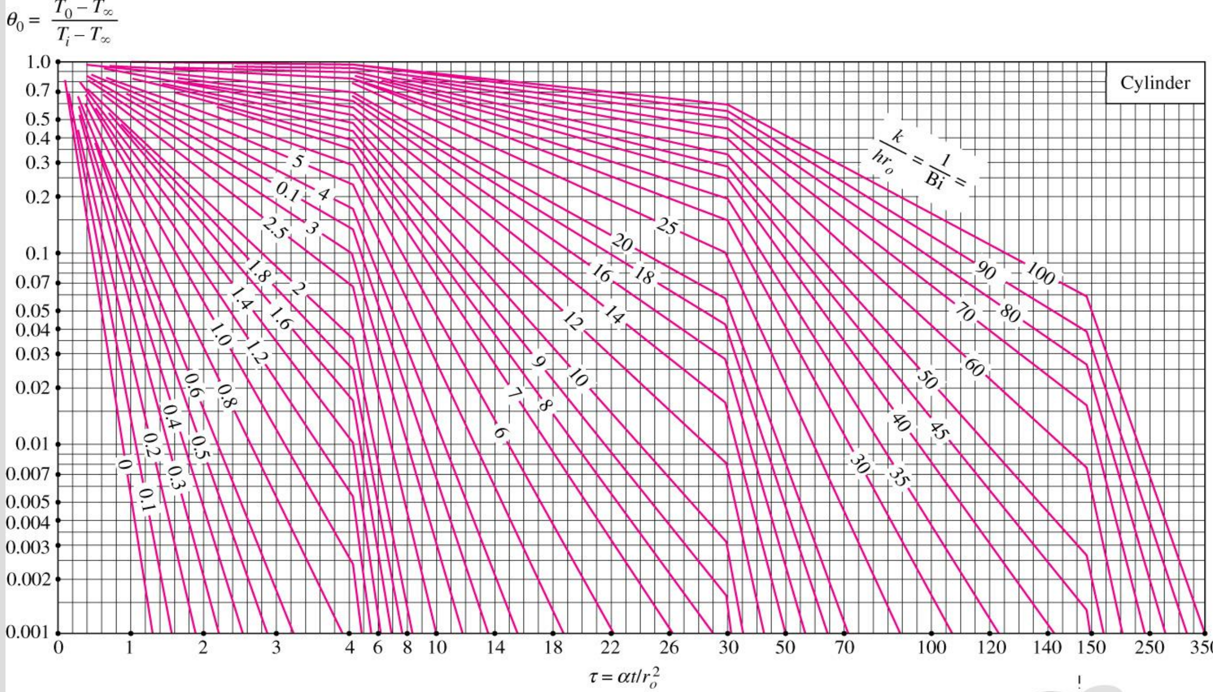


(*b*) Temperature distribution

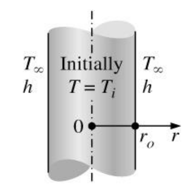


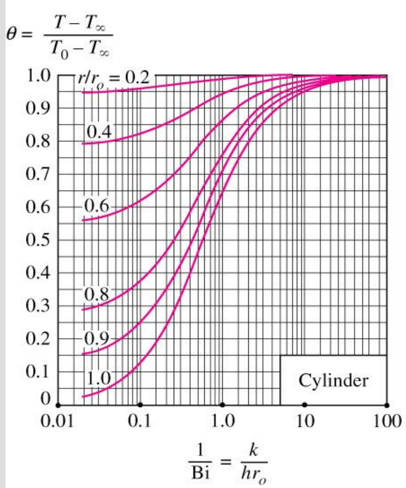
(*c*) Heat transfer



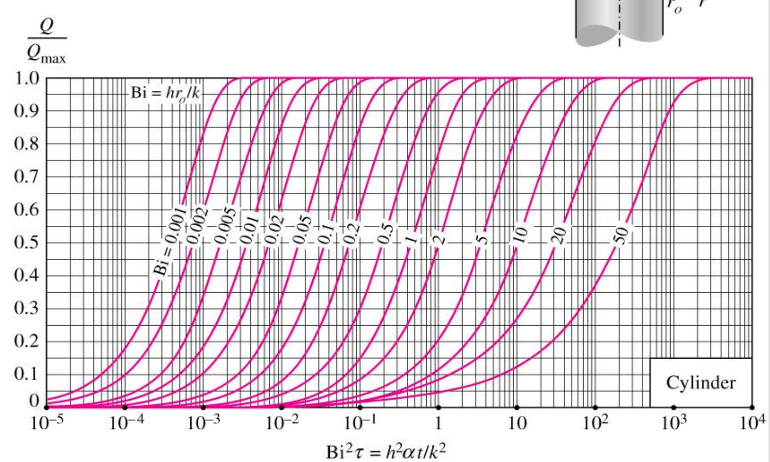


(*a*) Centerline temperature (from M. P. Heisler, "Temperature Charts for Induction and Constant Temperature Heating," *Trans. ASME 69*, 1947, pp. 227–36. Reprinted by permission of ASME International.)

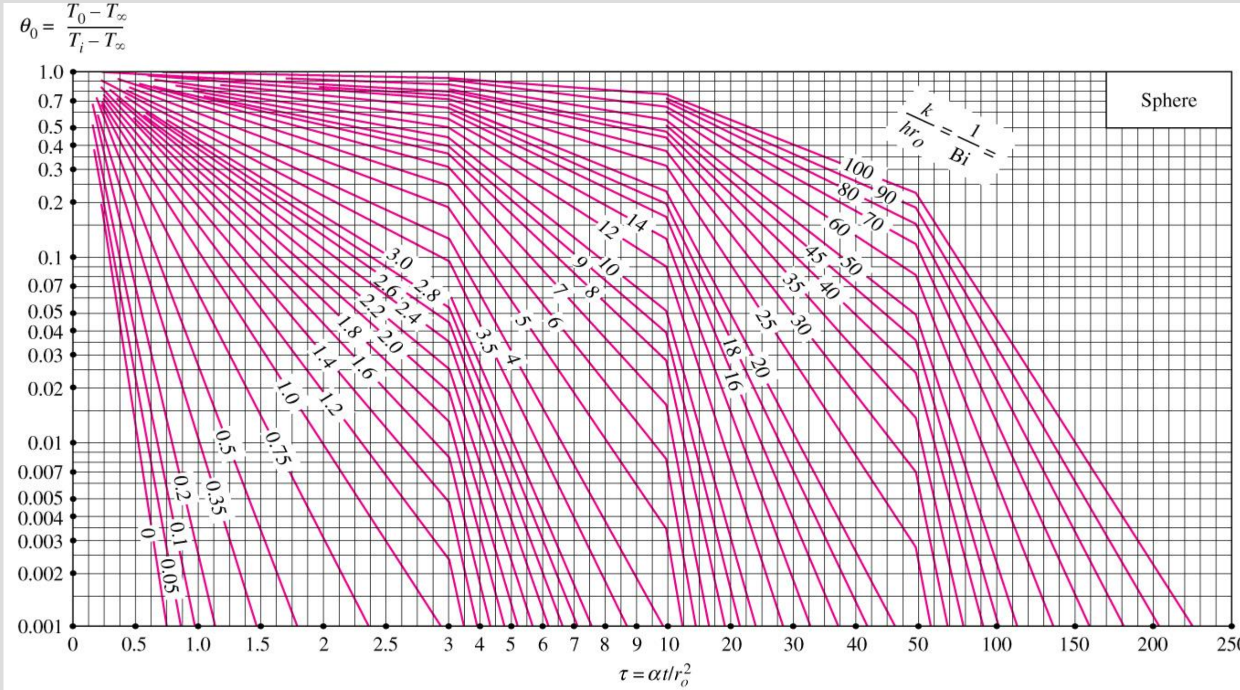




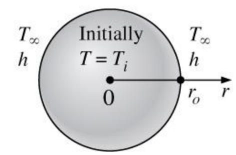
(*b*) Temperature distribution (from M. P. Heisler, "Temperature Charts for Induction and Constant Temperature Heating," *Trans. ASME 69*, 1947, pp. 227–36. Reprinted by permission of ASME International.)

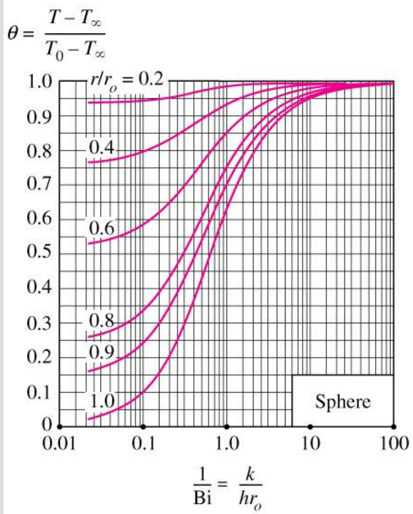


(*c*) Heat transfer (from H. Gröber et al.)

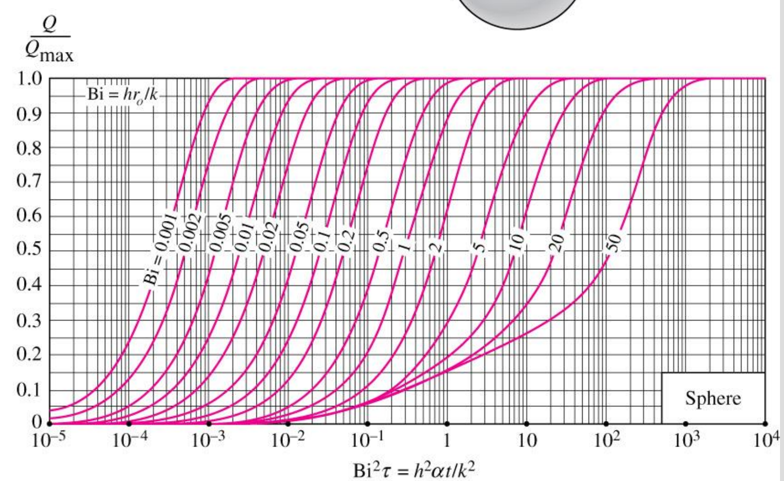


(*a*) Midpoint temperature (from M. P. Heisler, "Temperature Charts for Induction and Constant Temperature Heating," *Trans. ASME 69*, 1947, pp. 227–36. Reprinted by permission of ASME International.)





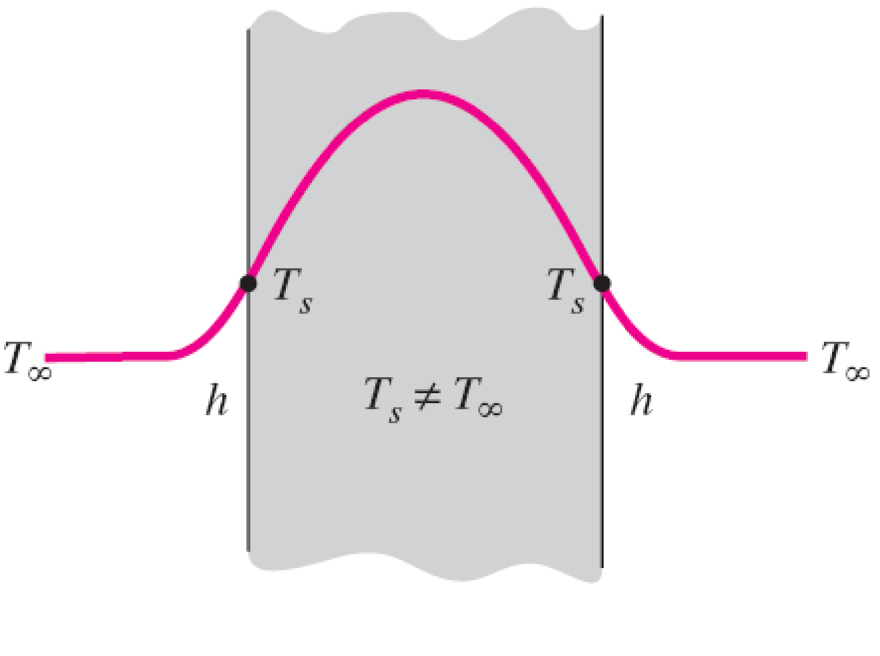
(*b*) Temperature distribution (from M. P. Heisler, "Temperature Charts for Induction and Constant Temperature Heating," *Trans. ASME 69*, 1947,



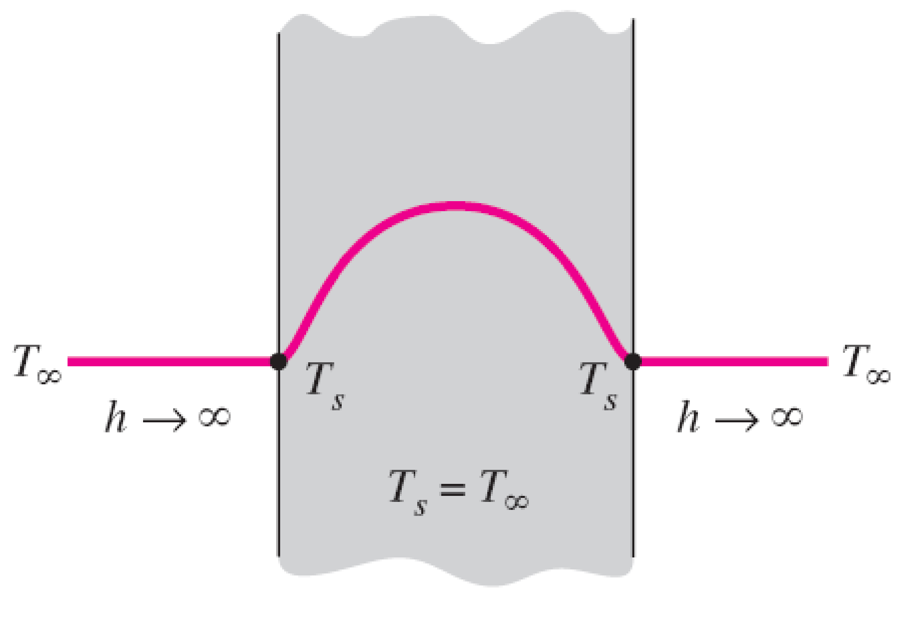
(*c*) Heat transfer (from H. Gröber et al.)

The dimensionless temperatures anywhere in a plane wall, cylinder, and sphere are related to the center temperature by

$$\frac{\theta_{\text{wall}}}{\theta_{0,\text{ wall}}} = \cos\left(\frac{\lambda_1 x}{L}\right), \quad \frac{\theta_{\text{cyl}}}{\theta_{0,\text{ cyl}}} = J_0\left(\frac{\lambda_1 r}{r_o}\right), \quad \text{and} \quad \frac{\theta_{\text{sph}}}{\theta_{0,\text{ sph}}} = \frac{\sin(\lambda_1 r/r_o)}{\lambda_1 r/r_o}$$



(a) Finite convection coefficient



(b) Infinite convection coefficient

The specified surface temperature corresponds to the case of convection to an environment at T_∞ with with a convection coefficient h that is *infinite.*

$$Q_{max} = mc_p(T_\infty - T_i) = \rho V c_p(T_\infty - T_i) \qquad \text{(kJ)}$$

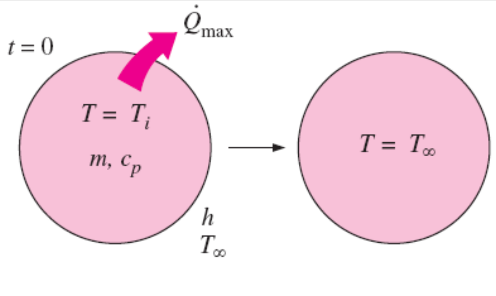
Plane wall: $$\left(\frac{Q}{Q_{max}}\right)_{wall} = 1 - \theta_{0,\,wall}\frac{\sin\lambda_1}{\lambda_1}$$

Cylinder: $$\left(\frac{Q}{Q_{max}}\right)_{cyl} = 1 - 2\theta_{0,\,cyl}\frac{J_1(\lambda_1)}{\lambda_1}$$

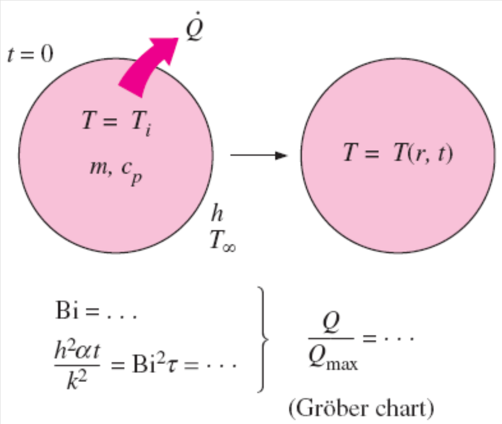
Sphere: $$\left(\frac{Q}{Q_{max}}\right)_{sph} = 1 - 3\theta_{0,\,sph}\frac{\sin\lambda_1 - \lambda_1\cos\lambda_1}{\lambda_1^3}$$

The fraction of total heat transfer Q/Q_{max} up to a specified time *t* is determined using the Gröber charts.

The use of the Heisler/Gröber charts and the one-term solutions already discussed is limited to the conditions specified at the beginning of this section: the body is initially at a *uniform* temperature, the temperature of the medium surrounding the body and the convection heat transfer coefficient are *constant* and *uniform*, and there is no *energy generation* in the body.



(*a*) Maximum heat transfer ($t \to \infty$)



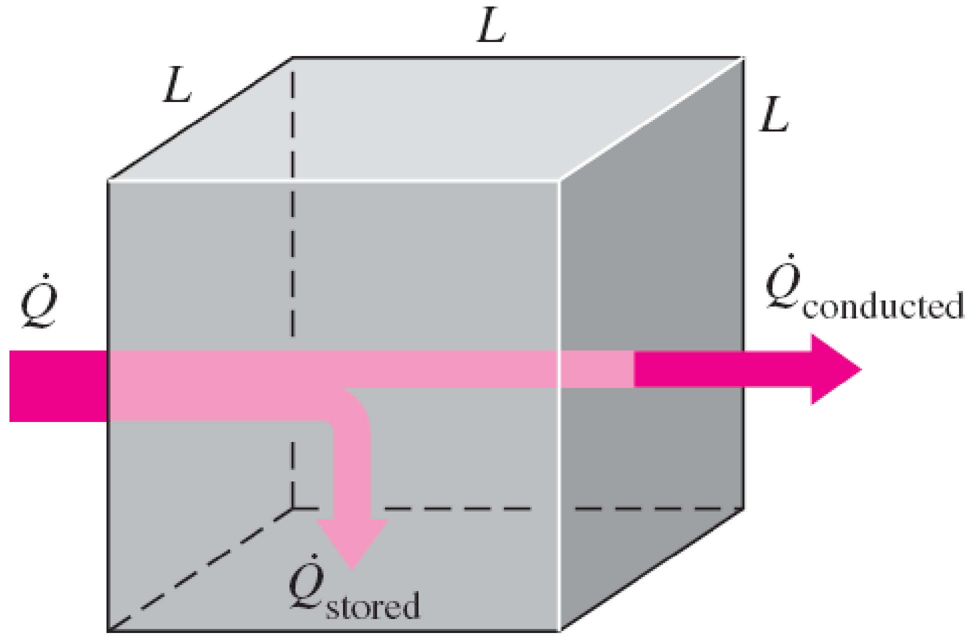
(*b*) Actual heat transfer for time *t*

The physical significance of the *Fourier number*

$$\tau = \frac{\alpha t}{L^2} = \frac{kL^2\,(1/L)}{\rho c_p L^3/t}\,\frac{\Delta T}{\Delta T} = \frac{\text{The rate at which heat is } \textit{conducted} \text{ across } L \text{ of a body of volume } L^3}{\text{The rate at which heat is } \textit{stored} \text{ in a body of volume } L^3}$$

- The Fourier number is a measure of *heat conducted* through a body relative to *heat stored.*
- A large value of the Fourier number indicates faster propagation of heat through a body.

Fourier number at time *t* can be viewed as the ratio of the rate of heat conducted to the rate of heat stored at that time.



$$\text{Fourier number: } \tau = \frac{\alpha t}{L^2} = \frac{\dot{Q}_{\text{conducted}}}{\dot{Q}_{\text{stored}}}$$

Example:

An ordinary egg can be approximated as a 5-cm-diameter sphere (Fig. 4–19). The egg is initially at a uniform temperature of 5°C and is dropped into boiling water at 95°C. Taking the convection heat transfer coefficient to be $h = 1200\ \text{W/m}^2 \cdot {}^\circ\text{C}$, determine how long it will take for the center of the egg to reach 70°C.

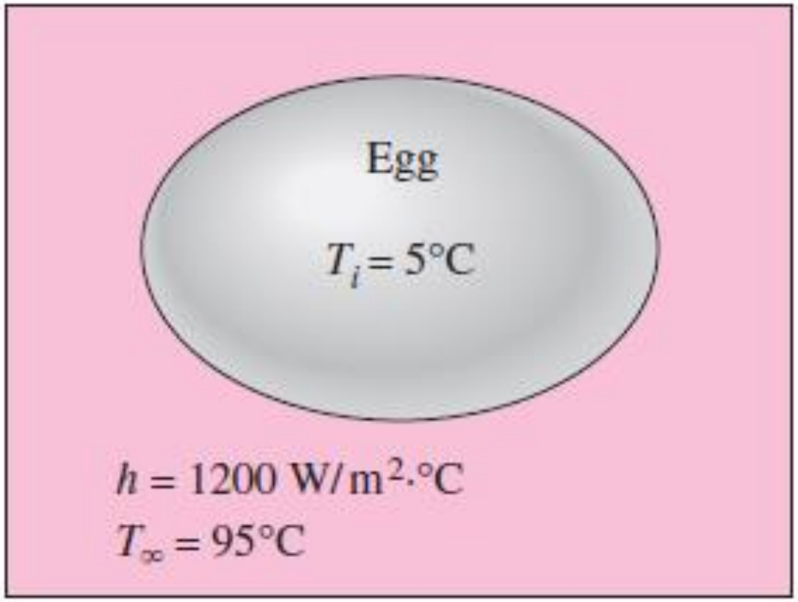
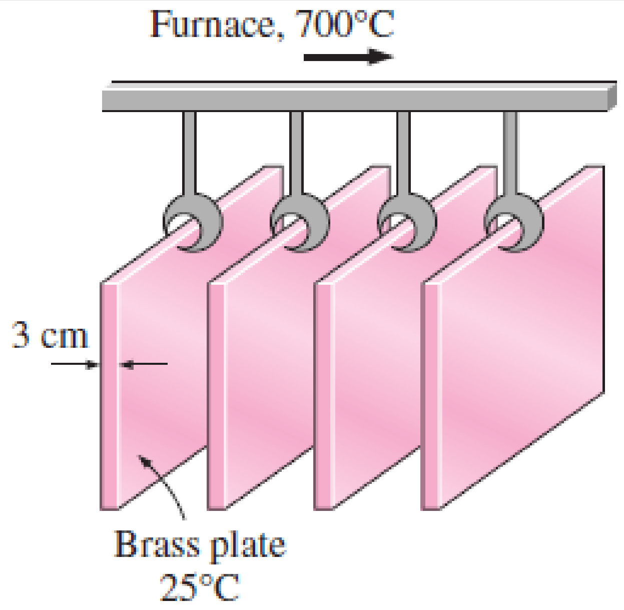


FIGURE 4–19

Example:

In a production facility, 3-cm-thick large brass plates (k = 110 W/m · °C, ρ = 8530 kg/m³, C_p = 380 J/kg · °C, and $\alpha = 33.9 \times 10^{-6}$ m²/s) that are initially at a uniform temperature of 25°C are heated by passing them through an oven maintained at 700°C. The plates remain in the oven for a period of 10 min. Taking the convection heat transfer coefficient to be h = 80 W/m² · °C, determine the surface temperature of the plates when they come out of the oven.



Example:

A long 20-cm-diameter cylindrical shaft made of stainless steel 304 comes out of an oven at a uniform temperature of 600°C (Fig. 4–21). The shaft is then allowed to cool slowly in an environment chamber at 200°C with an average heat transfer coefficient of $h = 80\ \text{W/m}^2 \cdot °\text{C}$. Determine the temperature at the center of the shaft 45 min after the start of the cooling process. Also, determine the heat transfer per unit length of the shaft during this time period.

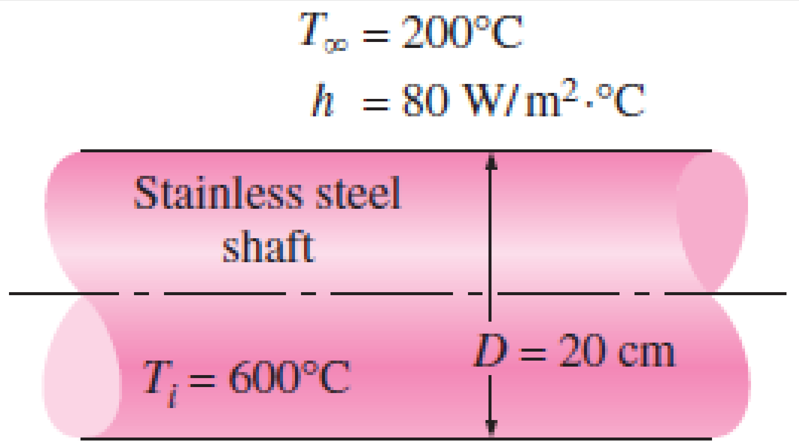
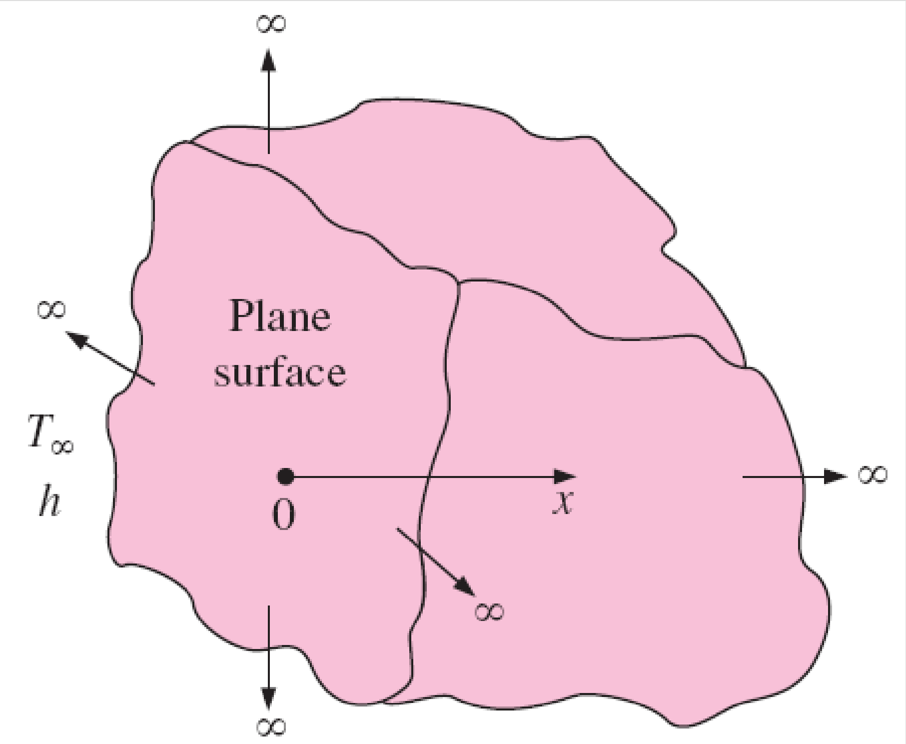


FIGURE 4–21

TRANSIENT HEAT CONDUCTION IN SEMI-INFINITE SOLIDS



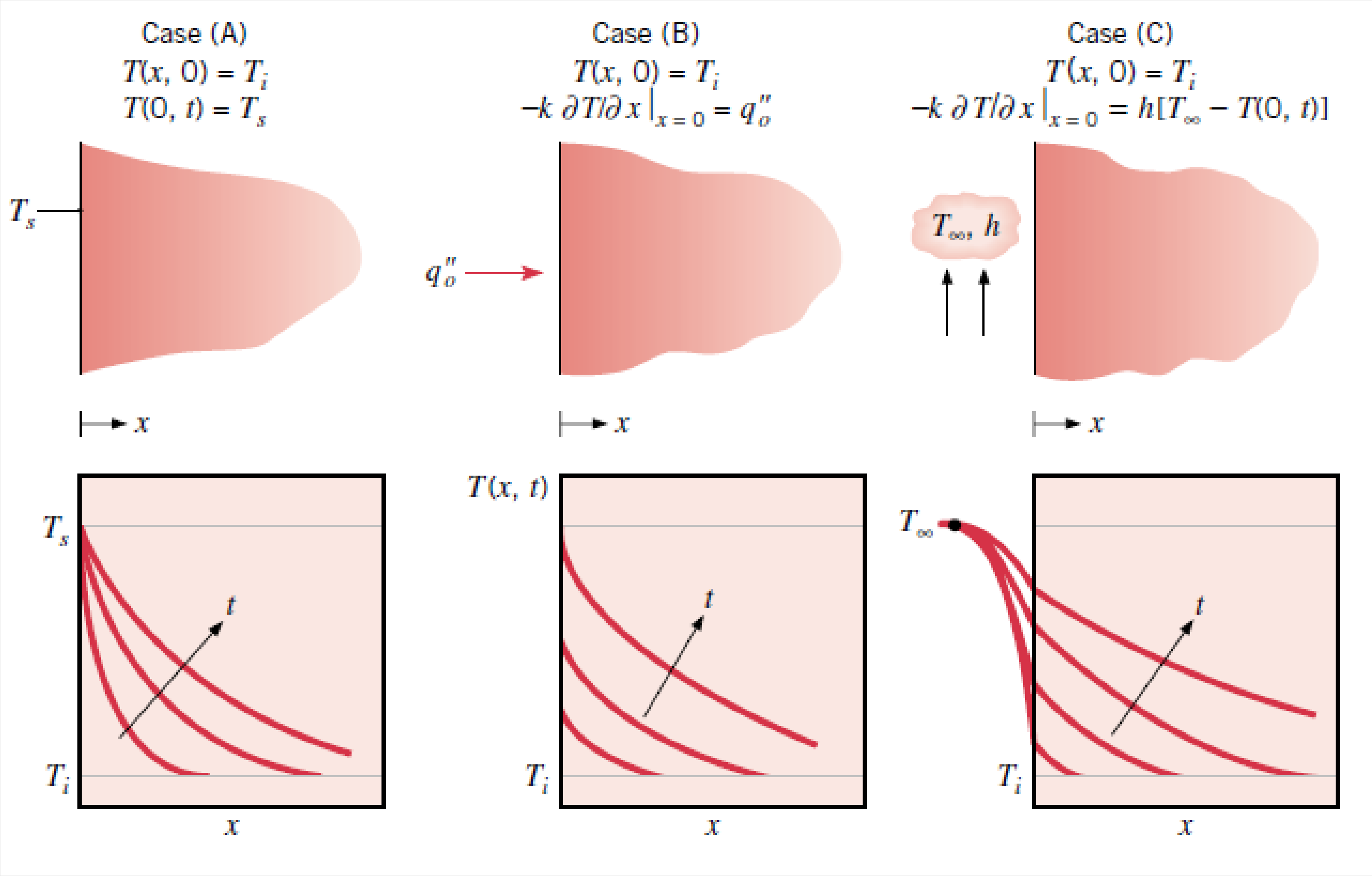
Schematic of a semi-infinite body.

Semi-infinite solid: An idealized body that has a *single plane surface* and extends to infinity in all directions.

The earth can be considered to be a semi-infinite medium in determining the variation of temperature near its surface.

A thick wall can be modeled as a semi-infinite medium if all we are interested in is the variation of temperature in the region near one of the surfaces, and the other surface is too far to have any impact on the region of interest during the time of observation.

For short periods of time, most bodies can be modeled as semi-infinite solids since heat does not have sufficient time to penetrate deep into the body.



Transient temperature distributions in a semi-infinite solid for three surfac conditions: constant surface temperature, constant surface heat flux, and surface convection

Analytical solution for the case of constant temperature T_s on the surface

Differential equation: $$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Boundary conditions: $T(0, t) = T_s$ and $T(x \to \infty, t) = T_i$

Initial condition: $T(x, 0) = T_i$

Similarity variable: $$\eta = \frac{x}{\sqrt{4\alpha t}}$$

$$\frac{d^2 T}{d\eta^2} = -2\eta \frac{dT}{d\eta}$$

$T(0) = T_s$ and $T(\eta \to \infty) = T_i$

$$\frac{T - T_s}{T_i - T_s} = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-u^2} du = \text{erf}(\eta) = 1 - \text{erfc}(\eta)$$

$$\text{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-u^2} du$$ error function

$$\text{erfc}(\eta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta \varepsilon^{-u^2} du$$ complementary error function

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \text{and} \quad \eta = \frac{x}{\sqrt{4\alpha t}}$$

$$\frac{\partial T}{\partial t} = \frac{dT}{d\eta}\frac{\partial \eta}{\partial t} = \frac{x}{2t\sqrt{4\alpha t}}\frac{dT}{d\eta}$$

$$\frac{\partial T}{\partial x} = \frac{dT}{d\eta}\frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{4\alpha t}}\frac{dT}{d\eta}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{d}{d\eta}\left(\frac{\partial T}{\partial x}\right)\frac{\partial \eta}{\partial x} = \frac{1}{4\alpha t}\frac{d^2 T}{d\eta^2}$$

Transformation of variables in the derivatives of the heat conduction equation by the use of chain rule.

Analytical solutions for different boundary conditions on the surface

Case 1: Specified Surface Temperature, T_s = constant

$$\frac{T(x, t) - T_i}{T_s - T_i} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \quad \text{and} \quad \dot{q}_s(t) = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}$$

Case 2: Specified Surface Heat Flux, $\dot{q}_s$ = constant.

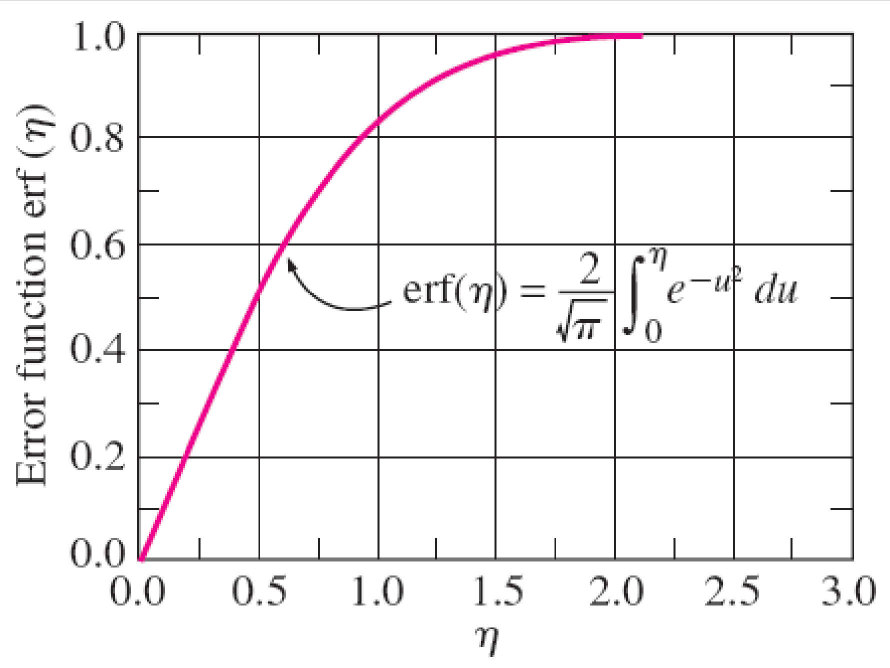
$$T(x, t) - T_i = \frac{\dot{q}_s}{k}\left[\sqrt{\frac{4\alpha t}{\pi}}\exp\left(-\frac{x^2}{4\alpha t}\right) - x\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)\right]$$

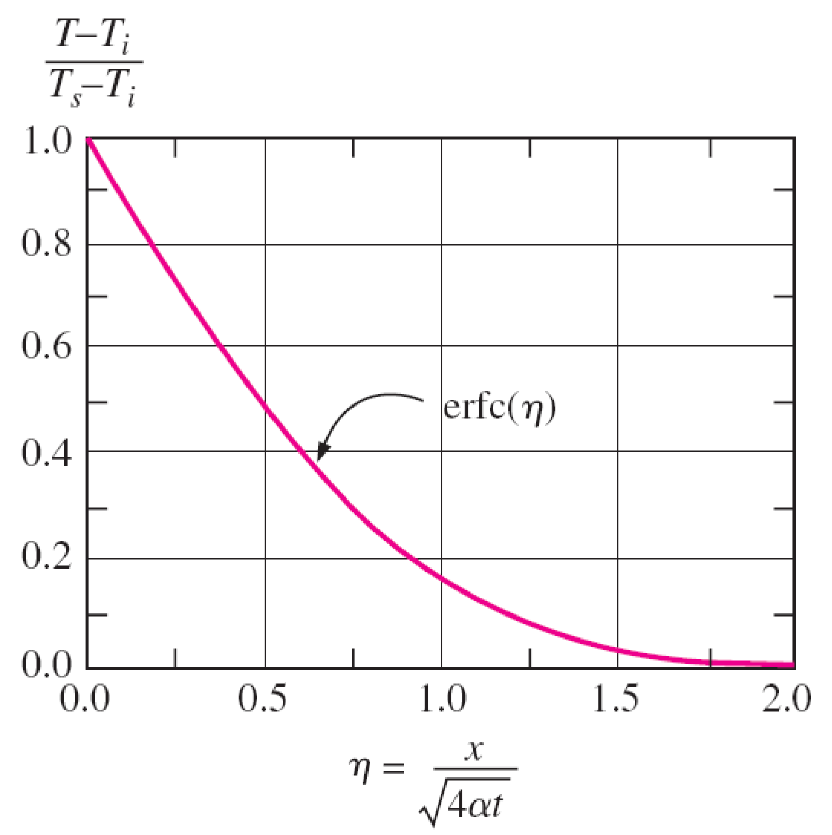
Case 3: Convection on the Surface, $\dot{q}_s(t) = h[T_\infty - T(0, t)]$.

$$\frac{T(x,t) - T_i}{T_\infty - T_i} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \exp\left(\frac{hx}{k} + \frac{h^2\alpha t}{k^2}\right)\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)$$

Case 4: Energy Pulse at Surface, e_s = constant.

$$T(x, t) - T_i = \frac{e_s}{k\sqrt{\pi t/\alpha}}\exp\left(-\frac{x^2}{4\alpha t}\right)$$

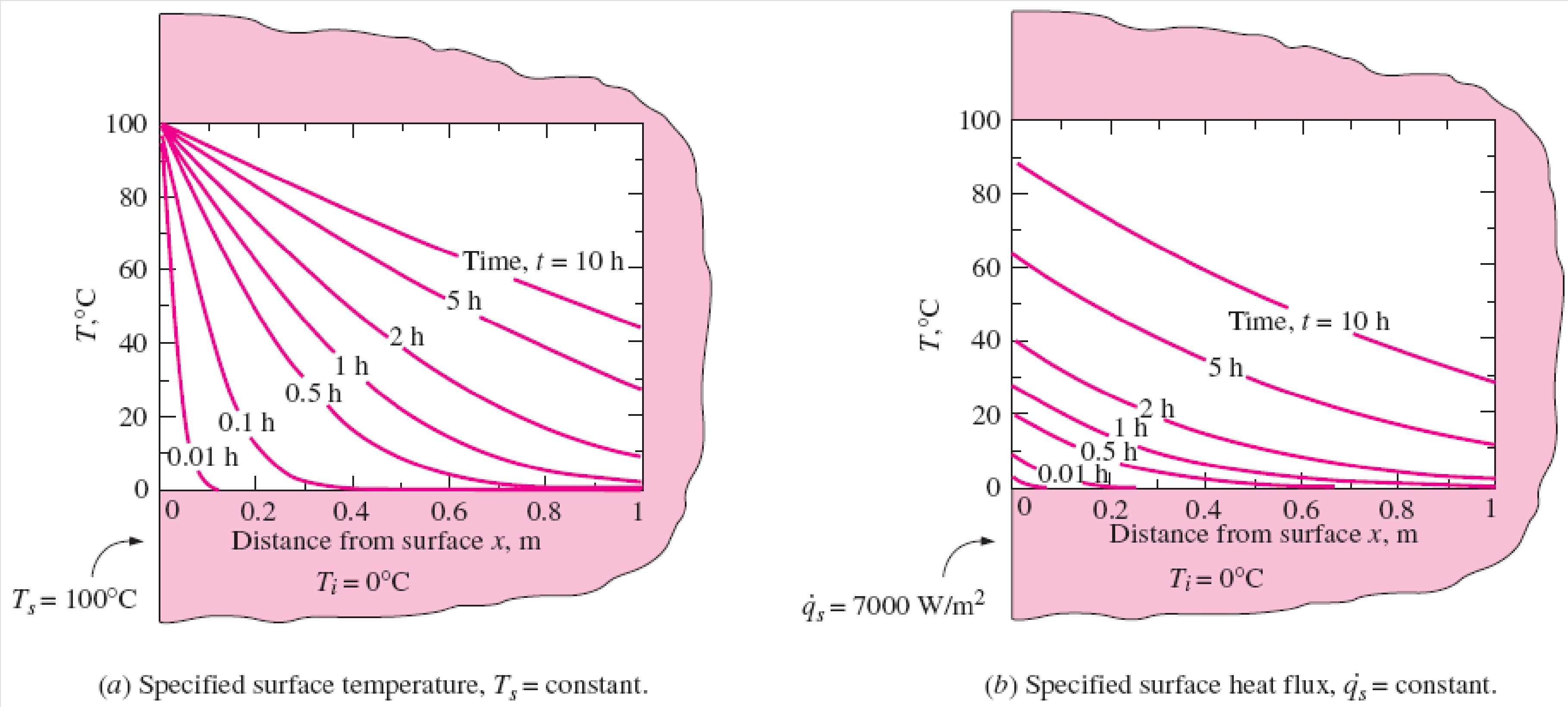




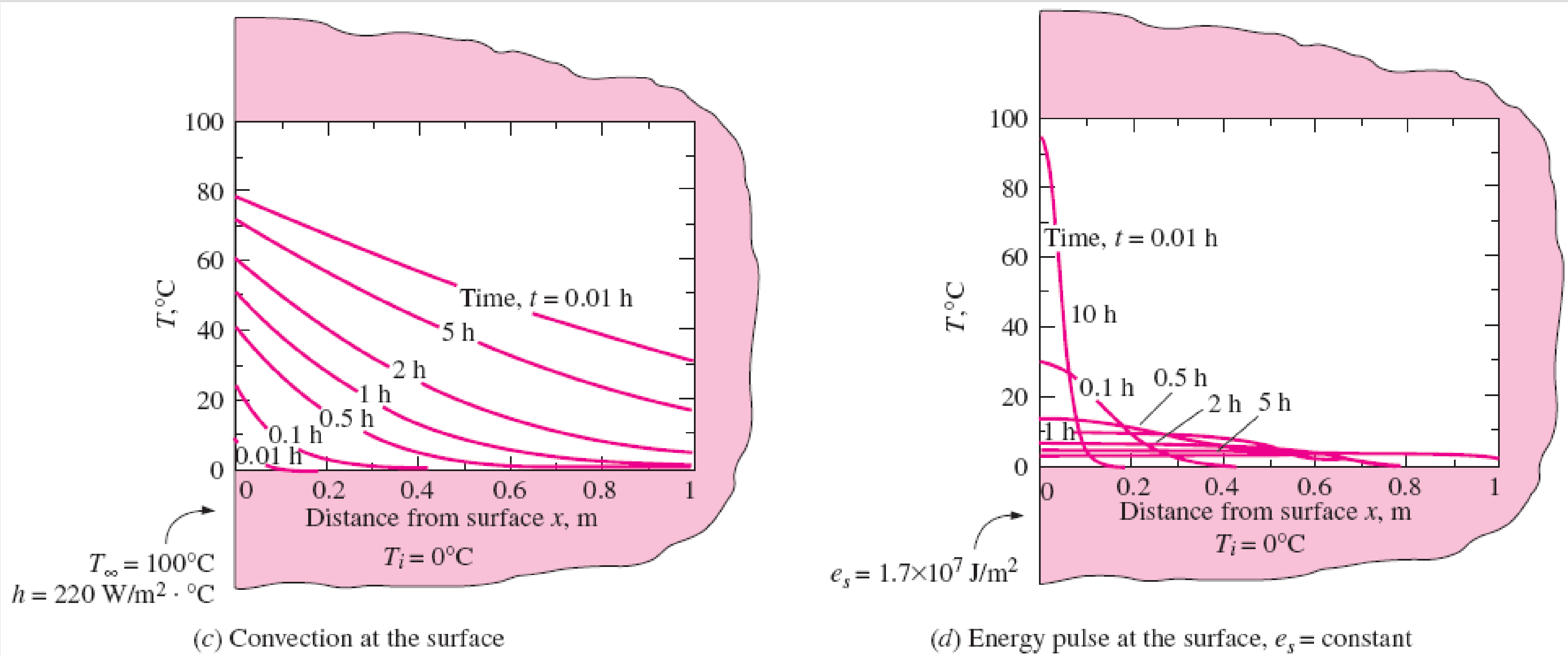
Error function is a standard mathematical function, just like the sine and cosine functions, whose value varies between 0 and 1.

The complementary error function

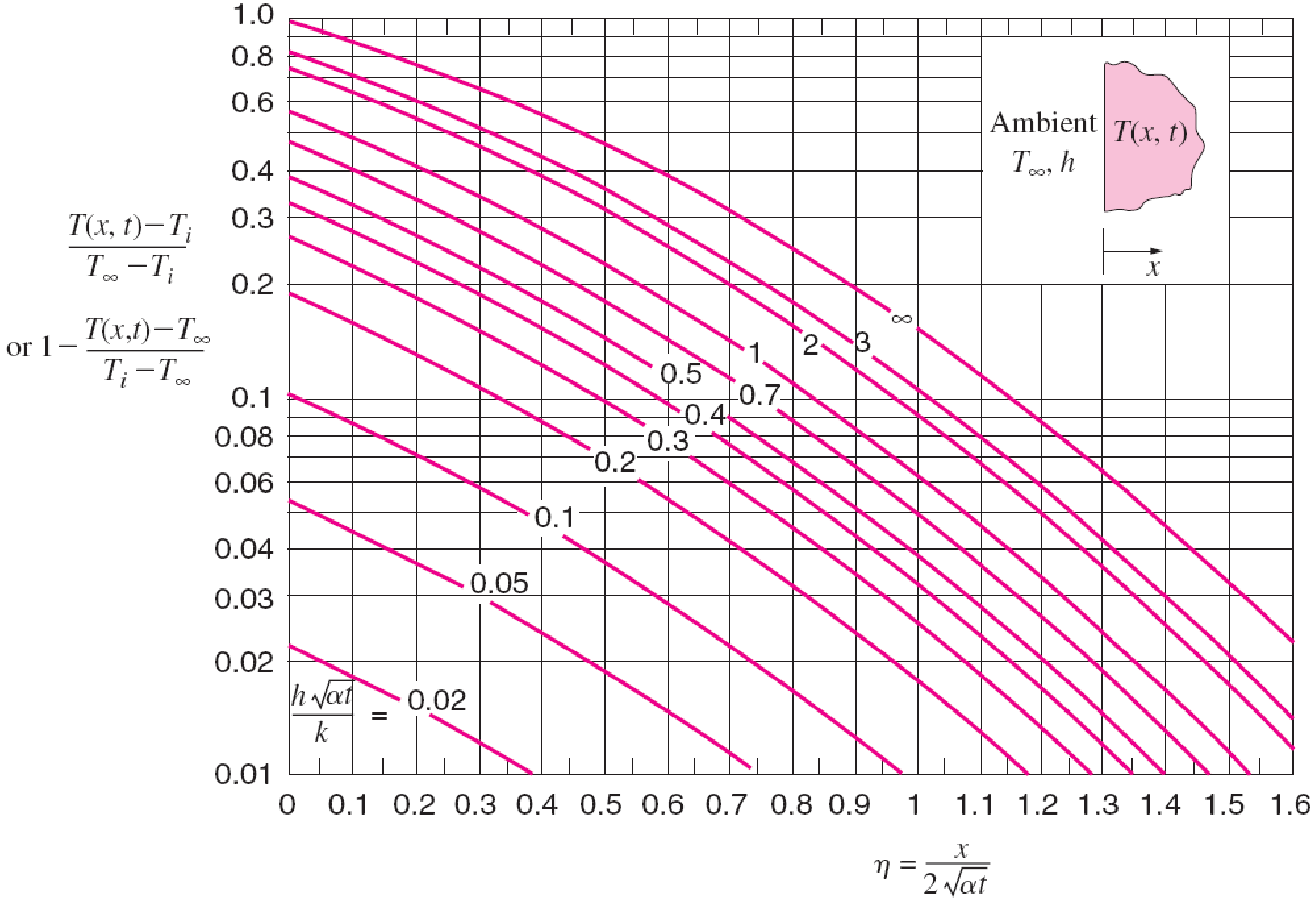
ξ	erfc (ξ)	ξ	erfc (ξ)	ξ	erfc (ξ)	ξ	erfc (ξ)	ξ	erfc (ξ)	ξ	erfc (ξ)
0.00	1.00000	0.38	0.5910	0.76	0.2825	1.14	0.1069	1.52	0.03159	1.90	0.00721
0.02	0.9774	0.40	0.5716	0.78	0.2700	1.16	0.10090	1.54	0.02941	1.92	0.00662
0.04	0.9549	0.42	0.5525	0.80	0.2579	1.18	0.09516	1.56	0.02737	1.94	0.00608
0.06	0.9324	0.44	0.5338	0.82	0.2462	1.20	0.08969	1.58	0.02545	1.96	0.00557
0.08	0.9099	0.46	0.5153	0.84	0.2349	1.22	0.08447	1.60	0.02365	1.98	0.00511
0.10	0.8875	0.48	0.4973	0.86	0.2239	1.24	0.07950	1.62	0.02196	2.00	0.00468
0.12	0.8652	0.50	0.4795	0.88	0.2133	1.26	0.07476	1.64	0.02038	2.10	0.00298
0.14	0.8431	0.52	0.4621	0.90	0.2031	1.28	0.07027	1.66	0.01890	2.20	0.00186
0.16	0.8210	0.54	0.4451	0.92	0.1932	1.30	0.06599	1.68	0.01751	2.30	0.00114
0.18	0.7991	0.56	0.4284	0.94	0.1837	1.32	0.06194	1.70	0.01612	2.40	0.00069
0.20	0.7773	0.58	0.4121	0.96	0.1746	1.34	0.05809	1.72	0.01500	2.50	0.00041
0.22	0.7557	0.60	0.3961	0.98	0.1658	1.36	0.05444	1.74	0.01387	2.60	0.00024
0.24	0.7343	0.62	0.3806	1.00	0.1573	1.38	0.05098	1.76	0.01281	2.70	0.00013
0.26	0.7131	0.64	0.3654	1.02	0.1492	1.40	0.04772	1.78	0.01183	2.80	0.00008
0.28	0.6921	0.66	0.3506	1.04	0.1413	1.42	0.04462	1.80	0.01091	2.90	0.00004
0.30	0.6714	0.68	0.3362	1.06	0.1339	1.44	0.04170	1.82	0.01006	3.00	0.00002
0.32	0.6509	0.70	0.3222	1.08	0.1267	1.46	0.03895	1.84	0.00926	3.20	0.00001
0.34	0.6306	0.72	0.3086	1.10	0.1198	1.48	0.03635	1.86	0.00853	3.40	0.00000
0.36	0.6107	0.74	0.2953	1.12	0.1132	1.50	0.03390	1.88	0.00784	3.60	0.00000



Variations of temperature with position and time in a large cast iron block ($\alpha = 2.31 \times 10^{-5}$ m^2/s, $k = 80.2$ W/m · °C) initially at 0 °C under different thermal conditions on the surface.



Variations of temperature with position and time in a large cast iron block ($\alpha = 2.31 \times 10^{-5}$ m²/s, $k = 80.2$ W/m · °C) initially at 0 °C under different thermal conditions on the surface.



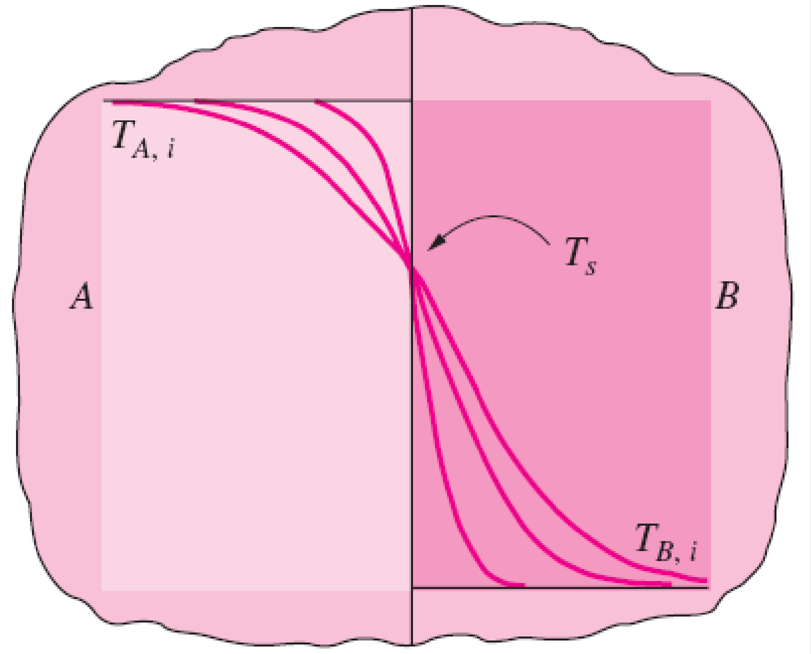
Variation of temperature with position and time in a semi-infinite solid initially at temperature T_i subjected to convection to an environment at T_∞ with a convection heat transfer coefficient of *h*.

Contact of Two Semi-Infinite Solids

When two large bodies *A* and *B*, initially at uniform temperatures $T_{A,i}$ and $T_{B,i}$ are brought into contact, they instantly achieve temperature equality at the contact surface.

If the two bodies are of the same material, the contact surface temperature is the arithmetic average, $T_s = (T_{A,i}+T_{B,i})/2$.

If the bodies are of different materials, the surface temperature T_s will be different than the arithmetic average.



Contact of two semi-infinite solids of different initial temperatures.

$$\dot{q}_{s,A} = \dot{q}_{s,B} \rightarrow -\frac{k_A(T_s - T_{A,i})}{\sqrt{\pi\alpha_A t}} = \frac{k_B(T_s - T_{B,i})}{\sqrt{\pi\alpha_B t}} \rightarrow \frac{T_{A,i} - T_s}{T_s - T_{B,i}} = \sqrt{\frac{(k\rho c_p)_B}{(k\rho c_p)_A}}$$

$$T_s = \frac{\sqrt{(k\rho c_p)_A}T_{A,i} + \sqrt{(k\rho c_p)_B}T_{B,i}}{\sqrt{(k\rho c_p)_A} + \sqrt{(k\rho c_p)_B}}$$

The interface temperature of two bodies brought into contact is dominated by the body with the larger $k\rho c_p$.

EXAMPLE: When a person with a skin temperature of 35°C touches an aluminum block and then a wood block both at 15°C, the contact surface temperature will be 15.9°C in the case of aluminum and 30°C in the case of wood.

Example:

In areas where the air temperature remains below 0°C for prolonged periods of time, the freezing of water in underground pipes is a major concern. Fortunately, the soil remains relatively warm during those periods, and it takes weeks for the subfreezing temperatures to reach the water mains in the ground. Thus, the soil effectively serves as an insulation to protect the water from subfreezing temperatures in winter.

The ground at a particular location is covered with snow pack at −10°C for a continuous period of three months, and the average soil properties at that location are $k = 0.4$ W/m · °C and $\alpha = 0.15 \times 10^{-6}$ m²/s (Fig. 4–24). Assuming an initial uniform temperature of 15°C for the ground, determine the minimum burial depth to prevent the water pipes from freezing.

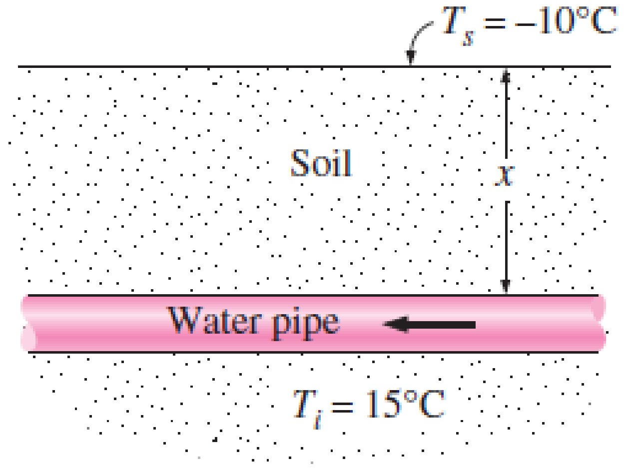
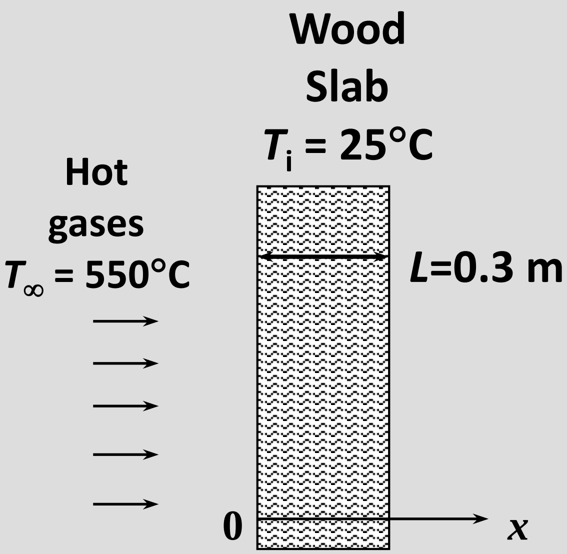


FIGURE 4–24

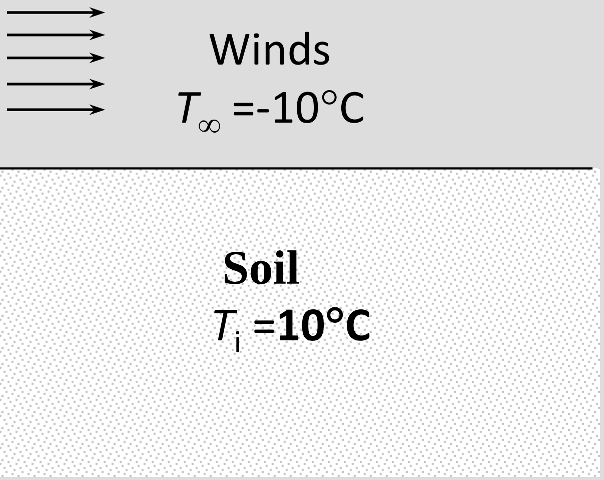
Example:

A thick wood slab (k = 0.17 W/m · °C and $\alpha = 1.28 \times 10^{-7}$ m^2/s) that is initially at a uniform temperature of 25°C is exposed to hot gases at 550°C for a period of 5 minutes. The heat transfer coefficient between the gases and the wood slab is 35 W/m^2 · °C. If the ignition temperature of the wood is 450°C, determine if the wood will ignite.



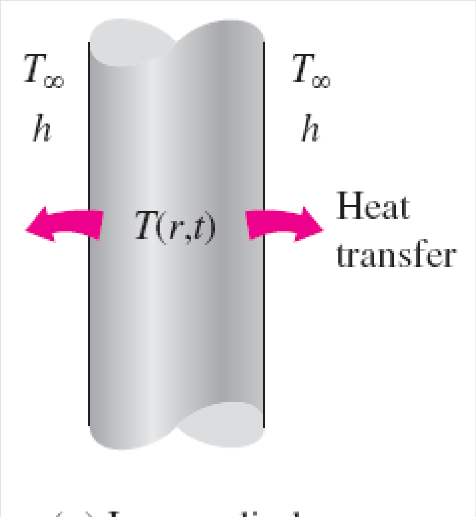
Example:

The soil temperature in the upper layers of the earth varies with the variations in the atmospheric conditions. Before a cold front moves in, the earth at a location is initially at a uniform temperature of 10°C. Then the area is subjected to a temperature of −10°C and high winds that resulted in a convection heat transfer coefficient of 40 W/m^2 · °C on the earth's surface for a period of 10 h. Taking the properties of the soil at that location to be k = 0.9 W/m · °C and α = 1.6 × 10^{-5} m^2/s, determine the soil temperature at distances 0, 10, 20, and 50 cm from the earth's surface at the end of this 10-h period.

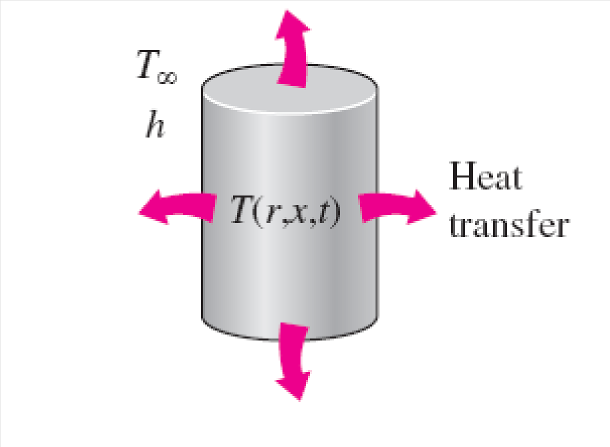


TRANSIENT HEAT CONDUCTION IN MULTIDIMENSIONAL SYSTEMS

- Using a superposition approach called the **product solution**, the transient temperature charts and solutions can be used to construct solutions for the *two-dimensional* and *three-dimensional* transient heat conduction problems encountered in geometries such as a short cylinder, a long rectangular bar, a rectangular prism or a semi-infinite rectangular bar, provided that *all* surfaces of the solid are subjected to convection to the *same* fluid at temperature T_∞, with the *same* heat transfer coefficient h, and the body involves no heat generation.
- The solution in such multidimensional geometries can be expressed as the *product* of the solutions for the one-dimensional geometries whose intersection is the multidimensional geometry.



(a) Long cylinder



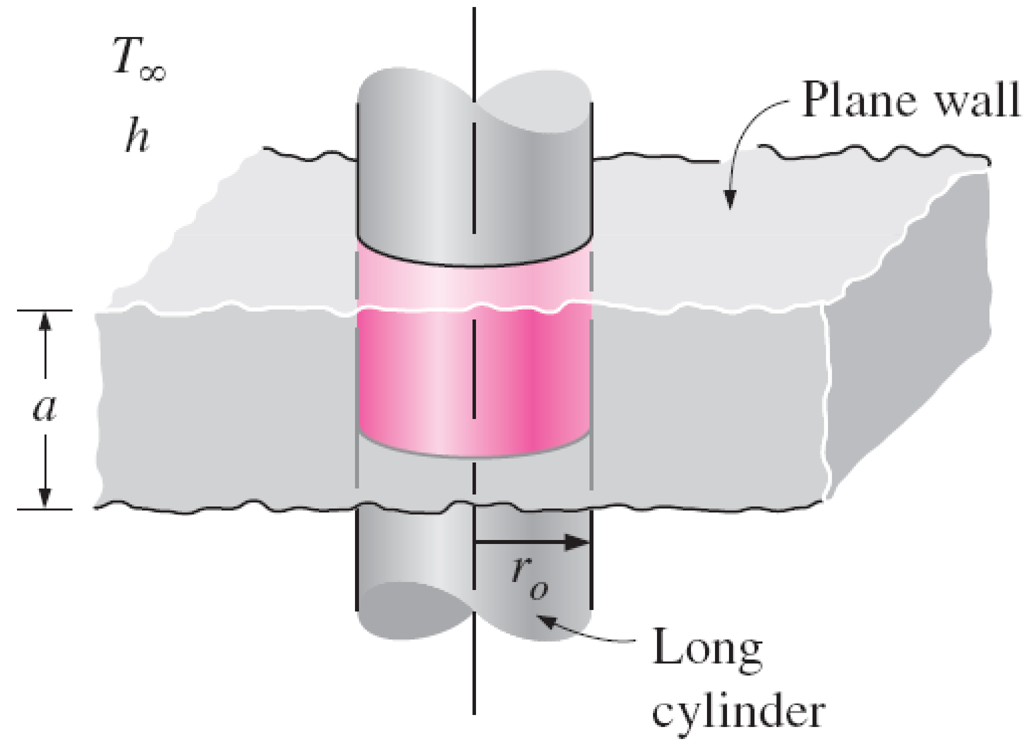
(b) Short cylinder (two-dimensional)

The temperature in a short cylinder exposed to convection from all surfaces varies in both the radial and axial directions, and thus heat is transferred in both directions.

The solution for a multidimensional geometry is the product of the solutions of the one-dimensional geometries whose intersection is the multidimensional body.

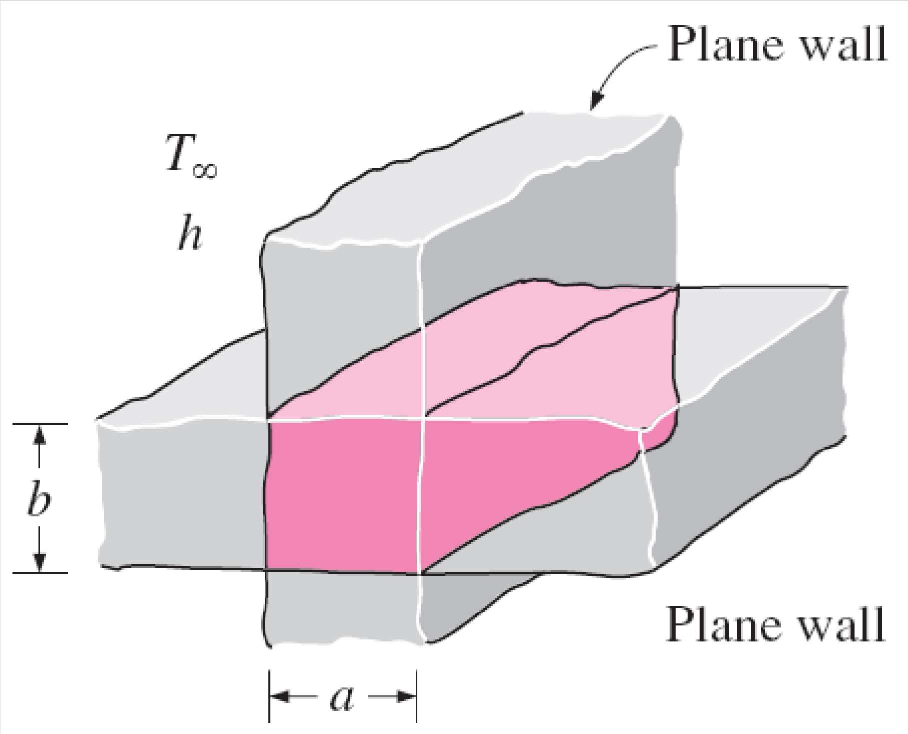
The solution for the two-dimensional short cylinder of height *a* and radius r_o is equal to the *product* of the nondimensionalized solutions for the one-dimensional plane wall of thickness *a* and the long cylinder of radius r_o.

$$\left(\frac{T(r,x,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{\text{short}\\\text{cylinder}}} = \left(\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{\text{plane}\\\text{wall}}} \left(\frac{T(r,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{\text{infinite}\\\text{cylinder}}}$$



A short cylinder of radius r_o and height *a* is the *intersection* of a long cylinder of radius r_o and a plane wall of thickness *a.*

$$\left(\frac{T(x, y, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{rectangular}\\ \text{bar}}} = \theta_{\text{wall}}(x, t)\theta_{\text{wall}}(y, t)$$



$$\theta_{\text{wall}}(x, t) = \left(\frac{T(x, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{plane}\\ \text{wall}}}$$

$$\theta_{\text{cyl}}(r, t) = \left(\frac{T(r, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{infinite}\\ \text{cylinder}}}$$

$$\theta_{\text{semi-inf}}(x, t) = \left(\frac{T(x, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{semi-infinite}\\ \text{solid}}}$$

A long solid bar of rectangular profile $a \times b$ is the *intersection* of two plane walls of thicknesses *a* and *b.*

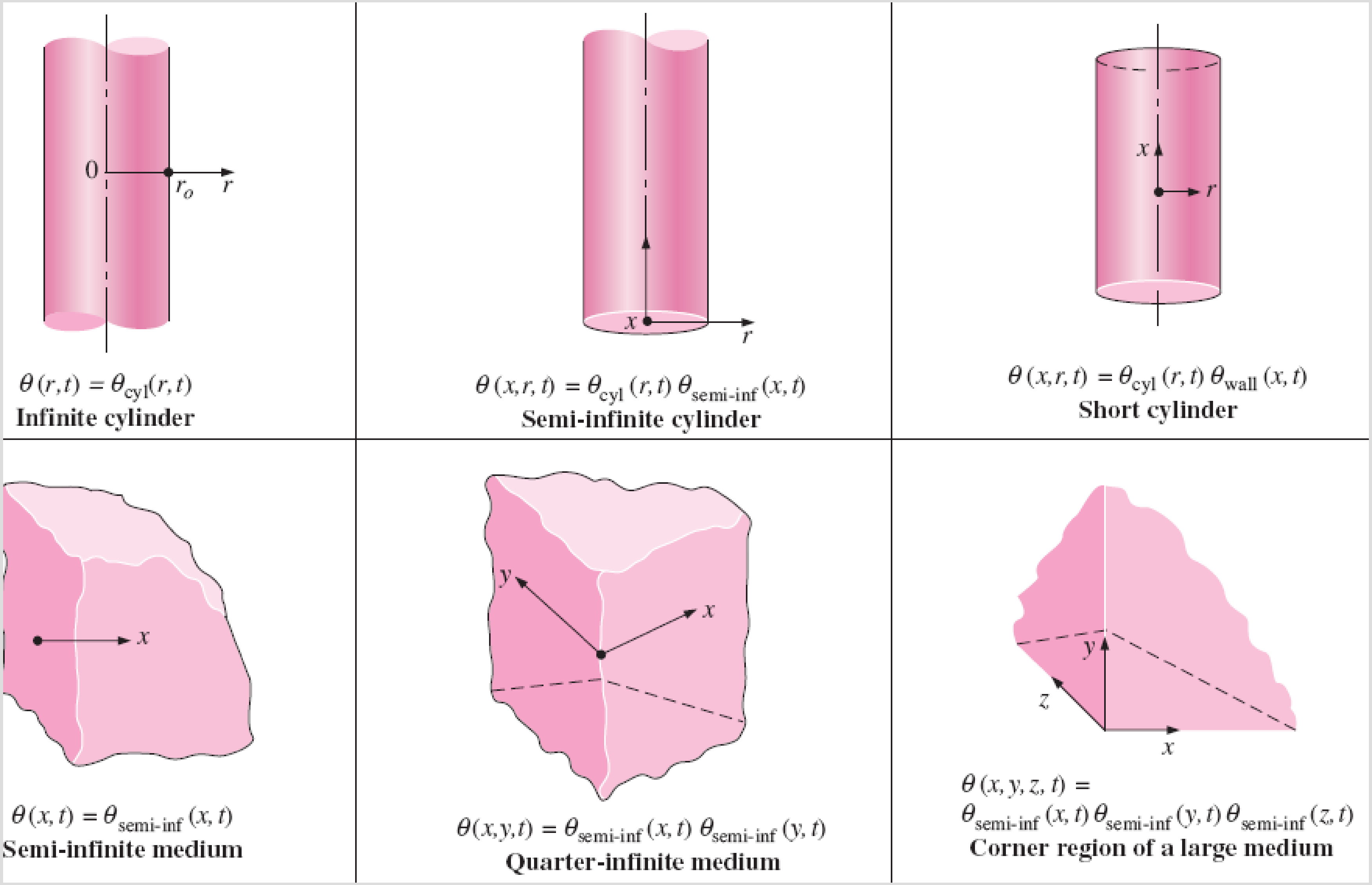
The transient heat transfer for a two-dimensional geometry formed by the intersection of two one-dimensional geometries 1 and 2 is

$$\left(\frac{Q}{Q_{\max}}\right)_{\text{total, 2D}} = \left(\frac{Q}{Q_{\max}}\right)_1 + \left(\frac{Q}{Q_{\max}}\right)_2 \left[1 - \left(\frac{Q}{Q_{\max}}\right)_1\right]$$

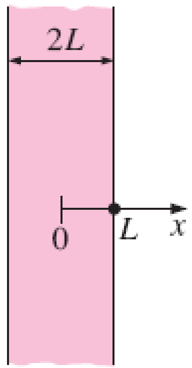
Transient heat transfer for a three-dimensional body formed by the intersection of three one-dimensional bodies 1, 2, and 3 is

$$\left(\frac{Q}{Q_{\max}}\right)_{\text{total, 3D}} = \left(\frac{Q}{Q_{\max}}\right)_1 + \left(\frac{Q}{Q_{\max}}\right)_2 \left[1 - \left(\frac{Q}{Q_{\max}}\right)_1\right] + \left(\frac{Q}{Q_{\max}}\right)_3 \left[1 - \left(\frac{Q}{Q_{\max}}\right)_1\right]\left[1 - \left(\frac{Q}{Q_{\max}}\right)_2\right]$$

Multidimensional solutions expressed as products of one-dimensional solutions for bodies that are initially at a uniform temperature T_i and exposed to convection from all surfaces to a medium at T_∞

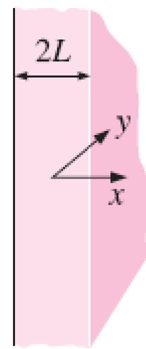


Multidimensional solutions expressed as products of one-dimensional solutions for bodies that are initially at a uniform temperature T_i and exposed to convection from all surfaces to a medium at T_∞



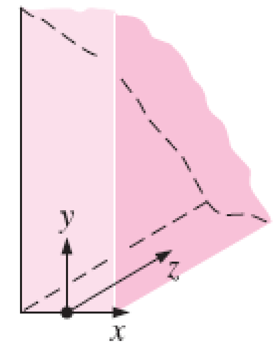
$\theta(x, t) = \theta_{wall}(x, t)$

Infinite plate (or plane wall)



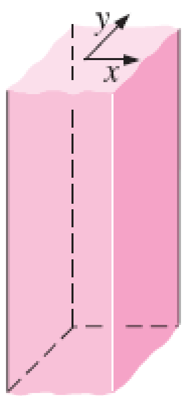
$\theta(x, y, t) = \theta_{wall}(x, t)\, \theta_{semi\text{-}inf}(y, t)$

Semi-infinite plate



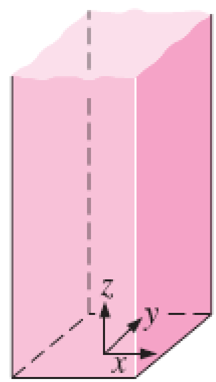
$\theta(x, y, z, t) =$
$\theta_{wall}(x, t)\, \theta_{semi\text{-}inf}(y, t)\, \theta_{semi\text{-}inf}(z, t)$

Quarter-infinite plate



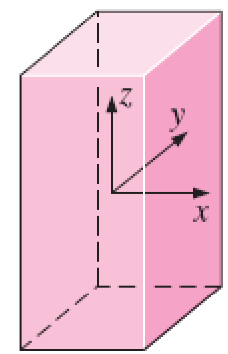
$\theta(x, y, t) = \theta_{wall}(x, t)\theta_{wall}(y, t)$

Infinite rectangular bar



$\theta(x,y,z, t) =$
$\theta_{wall}(x, t)\, \theta_{wall}(y, t)\, \theta_{semi\text{-}inf}(z, t)$

Semi-infinite rectangular bar



$\theta(x,y,z, t) =$
$\theta_{wall}(x, t)\theta_{wall}(y, t)\, \theta_{wall}(z, t)$

Rectangular parallelepiped

Summary

- Lumped System Analysis
  - ✓ Criteria for Lumped System Analysis
- Transient Heat Conduction in Large Plane Walls, Long Cylinders, and Spheres with Spatial Effects
  - ✓ Nondimensionalized One-Dimensional Transient Conduction Problem
  - ✓ Exact Solution of One-Dimensional Transient Conduction Problem
  - ✓ Approximate Analytical and Graphical Solutions
- Transient Heat Conduction in Semi-Infinite Solids
  - ✓ Contact of Two Semi-Infinite Solids
- Transient Heat Conduction in Multidimensional Systems

Fundamentals of Thermal-Fluid Sciences, 3rd Edition
Yunus A. Cengel, Robert H. Turner, John M. Cimbala
McGraw-Hill, 2008

NUMERICAL METHOD IN TRANSIENT HEAT CONDUCTION

Mehmet Kanoglu