BOUNCING BALL TRAJECTORY TRACKING USING INVERSE OPTIMAL CONTROL BY PASSIVITY: A HYBRID APPROACH

SEGUIMIENTO DE TRAYECTORIA DE PELOTA QUE ROBOTA USANDO CONTROL ÓPTIMO INVERSO POR PASIVIDAD: UN ENFOQUE HÍBRIDO

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Abstract: This article establishes a hybrid control strategy based on the theory of programmable automatons and the concepts of inverse optimal control to maintain the height of a bouncing ball system as an application case of vibro-impact systems. Hybrid modeling of the system in an ideal environment is obtained explicitly as the interaction between a continuous domain and a discrete domain. A control law based on the postulates of inverse optimal control by passivity is obtained to manipulate its post-impact dynamic behavior. The results are verified numerically using MATLAB®.

Keywords: Bouncing ball, hybrid dynamical system, trajectory tracking, inverse optimal control, vibro-impact system.

Resumen: Este artículo establece una estrategia de control híbrido basada en la teoría de los autómatas programables y los conceptos de control óptimo inverso para mantener la altura del sistema pelota que rebota como un caso de aplicación of sistemas de vibro-impacto. El modelado híbrido del sistema en un entorno ideal se obtiene explícitamente, como la interacción entre un dominio continuo y un dominio discreto. Se obtiene una ley de control basada en los postulados de control óptimo inverso por pasividad para manipular su comportamiento dinámico post-impacto para manipular el comportamiento dinámico del sistema, con el propósito de seguir una referencia específica después de cada impacto. Los resultados se verifican numéricamente usando MATLAB®.

Palabras clave: Pelota que rebota, sistema dinámico híbrido, seguimiento de trayectoria, control optimo inverso, sistema de vibro-impacto.

1. INTRODUCTION

Mechanical fault due to breakage in dynamic systems are often induced by the presence of vibrations caused by imbalances in rotating systems, shaft misalignment, the collision between components, misalignment of couplings, wear of parts and bearings, as well as aerodynamic and hydraulic stresses. It should be noted that these systems have unwanted nonlinear dynamic behaviors such as dead bands, friction, saturations, delays and discontinuities, making its analysis difficult through conventional methods (Xue and Fan, 2018). However, they can be represented by the theory of hybrid dynamic systems, capable to cover a wide number of phenomena, ranging from natural sciences to applied engineering (Naldi and Sanfelice, 2013). To reduce the damaging effect of unwanted collisions, algorithms have been developed aimed at conserving energy between friction-free solids subject to impacts, based on knowledge of the dynamic system trajectory in its transitory state (Deuflhard *et al.*, 2008). The fundamental problem of the bouncing ball has been widely used as a case study to analyze phenomena characteristic of vibration systems, where a ball under the effect of gravity is raised vertically by the action of a table with sinusoidal vibration (Okniński and Radziszewski, 2014). Therefore, the present work is focused on this particular type of system due to its dynamic richness, product of singularities during the collisions, which generate a discontinuous Poincaré map (Yue and Xie, 2013). Because of this discontinuous nature, it is sometimes pertinent to obtain the model using a hybrid approach, where the system can be conceptualized as a set of subsystems, through mathematical relations determined by constraints of the model itself, and operating environment. In this way, the hybrid system allows the application of specific controllers for each generated subsystem, which significantly reduces the complexity in both the design and the performance of the control action (Budd and Piiroinen, 2006; Flórez-Vargas and Alzate-Castaño, 2017).

Literature highlights some works related to the modeling and control of the bouncing ball. In (Kryzhevich and Wiercigroch, 2012) the grazing bifurcation phenomenon is considered to obtain a Newtonian model of the vibro-impact system due to the family of periodic solutions that this type of system presents, which are dependent on a parameter associated with the velocity component that vanishes with the bifurcation at the moment of impact. On the other hand, in (Leine and Heimsch, 2012) an extension of the Lyapunov stability method is proposed to guarantee the global convergence of a bouncing ball by means of an asymptotic attraction to finite time equilibrium, describing a convergent and exact solution. A predictive control model is applied in (Kulchenko and Todorov, 2011) to manipulate the robot locomotion acting on a bouncing ball, as an online optimal control problem where a finite-time prediction horizon is established that avoids the discontinuities in the dynamics during contact, in exchange to incorporate its effects into a cost functional with off-line adjustment. Also, in (Tian *et al., 2013*) a hybrid PID controller is designed, to guarantee the trajectory tracking on a ball that incorporates friction on its model, using a zero-crossing filter for the detection of the impact in a real-time implementation environment. As a potential application to high-temperature gas vibration analysis, in (Eshuis *et al.*, 2007) the vibrational dynamics of the bouncing ball are studied numerically due to its random nature, whose mathematical representation in the impact offers a special structure, where commuted trajectories and strange attractors can coexist.

In this work, as a contribution to the field of study of the control of vibro-impact systems, a novel alternative representation of bouncing ball as a hybrid dynamic system is proposed by defining sets of continuous dynamics and discrete events, based on the theory of finite automatons, avoiding to analyze the bifurcations produced during the collisions. Additionally, to achieve stabilization and trajectory tracking by means of passivation, an optimal inverse control is designed, where a cost functional conformed of the system information is minimized by a quadratic Lyapunov function used as a storage function, avoiding to solve the Hamilton-Jacobi-Bellman (HJB) equation (Sepulchre *et al.,* 2012). The content of the article is presented as follows: In section 2, the alternative hybrid formulation is introduced. Section 3 establishes inverse optimal control via passivation. Section 4 shows the hybrid representation of the bouncing ball, and the inverse optimal control law required to guarantee the desired trajectory tracking. The simulation results obtained using MATLAB® to numerically verify the performance of the proposed methodology are shown in section 5. Finally, this work is concluded

2. HYBRID DYNAMICAL SYSTEM: HDS

The approach of Hybrid Dynamic Systems has the versatility of representing a broad spectrum of systems, due to their ability to simultaneously describe phenomena with dominance in continuous-time and discrete-time, even when these singularities are in their dynamic behavior. From (Flórez-Vargas and Alzate-Castaño, 2017), a set-based formulation for the hybrid automaton is presented below.

In general, consider the dynamic system given by

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the n-dimensional vector of state variables, $f(\cdot) \in \mathbb{R}^n$ is a vector of not necessarily linear or smooth time functions, $\mathbf{u} \in \mathbb{R}^m$ is the m-dimensional control vector, and $\mathbf{x}_0 \in \mathbb{R}^n$ the initial conditions vector. Assume that the trajectory solution of equation (1) is given by the ordered pair $(t, \mathbf{x}(\mathbf{u}, t))$, which may belong either to the solution set of continuous dynamics C or to the set of discrete events D. The set C defined in equation (2) is expressed as the union of the l-dimensional region of solution of the dynamic equation $\dot{\mathbf{x}}$ belonging to the domain $\mathbb{R}^n$, whose dynamics are differentiable between *i-th* semi-open time intervals on the right $[t_i, t_{i+1})$, where t_{i+1} with $i = 1,2, \ldots, l-1$ correspond at the time in which continuity of differential equation is lost, and therefore fail to be differentiable.

$$C = \bigcup_{i=0}^{l-1} \left\{ \begin{array}{c} (t, \mathbf{x}(\mathbf{u},t)) \in [t_i, t_{i+1}) \times \mathbb{R}^n, \text{ such that} \\ \mathbf{x}(t):\ \dot{\mathbf{x}} = f(\mathbf{x},\mathbf{u},t), \quad \mathbf{x}(t_i) = \mathbf{x}_i \\ \wedge \\ t_{i+1} \in t \geq t_0 \,|\nexists\, \dot{\mathbf{x}}(t_{i+1}) \vee \dot{\mathbf{x}}(t_{i+1}) \to \infty \end{array} \right\} \quad (2)$$

On the other hand, the set D defined in equation (3), it is established as the union of $l-1$ events or singularities that occurs at the instant t_i with $i = 1,2, \ldots, l-1$, where the limit on the right of the differential equation solution is different from the limit on the left, which implies an indefinite limit; or simply the limit at time t_i is non-existent, as occurs in functions that describe vertices. In equation (3), the negative superscript of t_i^- refers to the infinitesimal instant of time occurring before the break of the system continuity $\dot{\mathbf{x}}$ at time t_i, while the positive superscript of t_i^+ represents the same at the infinitesimal instant of time after its rupture in t_i.

$$D = \bigcup_{i=1}^{l-1} \left\{ \begin{array}{c} (t, \mathbf{x}(\mathbf{u},t_i^+)) \in [t_i, t_{i+1}] \times \mathbb{R}^n, \text{ such that} \\ \lim_{t \to t_i^-} \mathbf{x}(\mathbf{u},t) \neq \lim_{t \to t_i^+} \mathbf{x}(\mathbf{u},t) \\ \vee \\ \nexists \lim_{t \to t_i} \dfrac{\mathbf{x}(\mathbf{u},t) - \mathbf{x}(\mathbf{u},t_i)}{t - t_i} \end{array} \right\} \quad (3)$$

In this way, the system of equation (1) is defined as hybrid through joining a set of continuous dynamic subsystems by means of a set of discrete events, as can be seen in equation (4). This interaction can be observed in Fig. 1, where two continuous dynamic subsystems with associated initial conditions are linked through the discrete event occurred at time t_1.

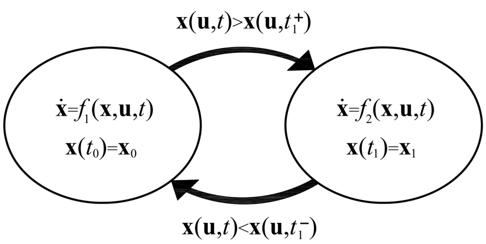

Fig. 1. Transition between subsystems

$$\mathbf{x}(\mathbf{u},t) = C \cup D, \forall t \in \mathbb{R}^+ . \quad (4)$$

3. INVERSE OPTIMAL CONTROL

3.1 Lie derivative

The Lie derivate of a smooth vector field $\mathrm{h}(\mathbf{x})$ along a smooth vector field $\mathrm{f}(\mathbf{x})$ is the vector field defined as

$$L_f h(\mathbf{x}) = \frac{\partial h(\mathbf{x})}{\partial \mathbf{x}} f(\mathbf{x}), \quad (5)$$

where the mappings $f(\mathbf{x})\colon \mathfrak{D} \to \mathbb{R}^n$, $h(\mathbf{x})\colon \mathfrak{D} \to \mathbb{R}^n$, in which $\mathfrak{D} \subset \mathbb{R}^n$ is the domain of the functions.

3.2 Optimal stabilizing control

Consider the dynamical system

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})u(\mathbf{x}), \quad (6)$$

where $\mathbf{x} \in \mathbb{R}^n$, and $\mathrm{u}(\mathbf{x})$ is a feedback control law. The problem of optimal control is to design a stabilizing control law, such that a certain cost function is minimized and asymptotic stability of the equilibrium point $\mathbf{x} = 0$ is achieved. A direct approach requires solving the Hamilton-Jacobi-Bellman (HJB) equation, which implies enormous mathematical complexity. From (Sepulchre *et al.,* 2012), the cost functional is given by

$$J = \int_0^\infty \left(l(\mathbf{x}) + u^T(\mathbf{x}) R(\mathbf{x}) u(\mathbf{x}) \right) dt, \quad (7)$$

where $l(\mathbf{x}) \geq 0$ and $R(\mathbf{x}) > 0$ for all $\mathbf{x}$. Thus, the value of $u(\mathbf{x})$ that minimizes (7) is called optimal and it is denoted as $u^*(\mathbf{x})$.

3.2 Optimality and stability

Suppose there exists a semi-definite positive function with a continuous first derivative, capable of satisfying the HJB equation given by

$$\begin{aligned} & l(\mathbf{x}) + L_f V(\mathbf{x}) \\ & -\frac{1}{4} L_g V(\mathbf{x}) R^{-1}(\mathbf{x}) \left(L_g V(\mathbf{x}) \right)^T = 0, \end{aligned} \quad (8)$$

with $V(0) = 0$, such that the asymptotic stability to the equilibrium $\mathbf{x} = 0$ is achieved if the feedback control is established as

$$u^*(\mathbf{x}) = -\frac{1}{2} R^{-1}(\mathbf{x}) \left(L_g V(\mathbf{x}) \right)^T (\mathbf{x}). \quad (9)$$

Therefore, $u^*(\mathbf{x})$ is the optimal stabilizing control that minimizes (6), ensuring that $\mathbf{x}$ tends to zero as time tends to infinity.

3.3 Inverse optimal design

In the inverse approach, the stabilizing control law that minimizes the functional cost is first designed, and posteriorly the $l(\mathbf{x}) \geq 0$ and $R(\mathbf{x}) > 0$ functions are chosen. In this way, the control law that solves the optimal inverse problem of system (5) is given by

$$u(\mathbf{x}) = \tilde{k}(\mathbf{x}) = -\frac{1}{2} R^{-1}(\mathbf{x}) \left(L_g V(\mathbf{x}) \right)^T (\mathbf{x}) \,,$$

where for a storage function selected as a definite positive control Lyapunov function $V(\mathbf{x})$, the negative semidefiniteness of $\dot{V}(\mathbf{x})$ is achieved if $u(\mathbf{x}) = \frac{1}{2}\tilde{k}(\mathbf{x})$, that is

$$\dot{V}(\mathbf{x}) = L_f V(\mathbf{x}) - \frac{1}{2} L_g V(\mathbf{x}) \tilde{k}(\mathbf{x}) \leq 0 \,. \quad (10)$$

Assuming the function $-l(\mathbf{x})$ as the right side of (10), such that

$$l(\mathbf{x}) = -L_f V(\mathbf{x}) + \frac{1}{2} L_g V(\mathbf{x}) \tilde{k}(\mathbf{x}) \geq 0 \,, \quad (11)$$

then $V(\mathbf{x})$ is solution of the HJB equation (8), which implies that (9) is the inverse optimal control law (Freeman and Kokotovic, 1993).

4. METHODOLOGY

Next, the concepts presented in the previous sections will be applied to obtain the alternative hybrid modeling of the bouncing ball vibro-impact type system, and an inverse optimal control law to guarantee the desired trajectory tracking to a preset reference with a minimum operating error.

4.1 Hybrid bouncing ball

Establishing the position of the ball with respect to the rigid surface as $x\,(t)$ and its velocity as $\dot{x}(t)$, there are two main scenarios of movement: the first one refers to the episodes in which the ball is in free motion with $x(t) > 0$ when the ground reference is at $x(t) = 0$. The second one occurs at moments when the ball collides with the surface at $x(t) = 0$. Fig. 2 illustrates the kinematics of the bouncing ball, establishing the position and sense of the velocity. Newton's second law is considered for free-fall of bodies in ideal conditions, where the acceleration of body depends exclusively on gravity acceleration, and therefore its kinematics are independent of object mass.

Establishing the position and velocity state variables as x_1 and x_2 respectively, the representation of the system in state space form in free-fall is obtained

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -a_g \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}. \quad (12)$$

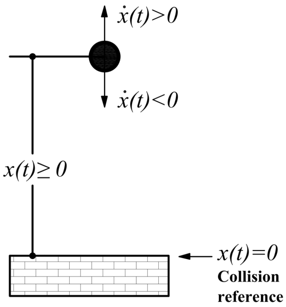

Fig. 2. Bouncing ball kinematic diagram

Therefore, the equations describing the kinematics in free-fall are given by

$$x_1(t) = -\frac{1}{2} a_g [t - t_0]^2 + v_0 [t - t_0] + x_0 \quad (13)$$

$$x_2(t) = -a_g [t - t_0] + v_0 \,, \quad (14)$$

where a_g is the gravity acceleration. On the other hand, when the ball collides against the rigid surface with a coefficient of restitution k (where k is a value between 0 and 1) at a reference height of x_1= 0, the position of the ball does not change instantly but the velocity does. Thus, assuming t_1 as the instant in which it collides for the first time with the ground, t_1^- as the infinitesimally inferior instant to t_1 where the ball reaches its maximum velocity in the descent, and t_1^+ as the infinitesimally superior moment of time to the impact on t_1, the velocity of the post-impact system is given by

$$x_2(t_1^+) = -k\, x_2(t_1^-). \quad (15)$$

So, based on equations (13) - (15), the kinematics of the bouncing ball system are defined, both along the free trajectory defined between bounces, and in the discrete instants in which the ball collides against the rigid surface, describing the position and velocity of the particle throughout $t \geq t_0$ as

$$x_1(t) > 0, \quad \left\{ \begin{array}{l} x_1(t) = -\frac{1}{2} a_g [\tau]^2 + v_0 [\tau] + x_0 \\ x_2(t) = -a_g [\tau] + v_0 \end{array} \right\} \quad (16)$$

$$x_1(t) = 0, \quad \left\{ \begin{array}{l} x_1(t) = 0 \\ x_2(t^+) = -k\, x_2(t^-) \end{array} \right\}, \quad (17)$$

where $\tau = t - t_0$. Additionally, to determine the instant time in which collisions occur, it is considered that the ball is thrown vertically upward with an initial velocity v_0 at the time t_0, and it is made to coincide with the value of the position in the impact $x_1(t) = 0$. In summary, the instants of time in which the dynamics of the system bifurcate by an abrupt change in the direction and magnitude of its velocity are determined as

$$t_i = \frac{2v_{i-1}}{a_g} + t_{i-1}, \quad i = 1,\ldots,N \tag{18}$$

where t_i is the instant of the *i-th* collision, and v_{i-1} is the *i-th* exit velocity, considering N impacts.

From equations (2), (3) and (18), and the vector representation of a free-fall system without friction of the air defined in equation (12), the continuous trajectory of the bouncing ball system is modeled through the set C given by

$$C = \cup_{i=0}^{l-1} \left\{ \begin{array}{c} (t, \mathbf{x}(u,t)) \in [t_i, t_{i+1}) \times \mathbb{R}^2, \text{ such that} \\ \mathbf{x}(t):\ \dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -a_g \end{bmatrix} \\ \mathbf{x}(t_i) = \mathbf{x}_i = \begin{bmatrix} x(t_i) \\ v(t_i) \end{bmatrix} \\ \wedge \\ \left(t_i = \frac{2v_{i-1}}{a_g} + t_{i-1} \right) \in t \geq t_1, \text{ such that} \\ \nexists \dot{\mathbf{x}}(t_{i+1}) \vee \dot{\mathbf{x}}(t_{i+1}) \to \infty \end{array} \right\} \tag{19}$$

where t_{i+1} is the moment in which the ball hits the rigid surface. On the other hand, in (20) the set of discrete events D is defined, which brings together the solutions of the system at the moment when discontinuities occur during times t_i.

$$D = \cup_{i=1}^{l-1} \left\{ \begin{array}{c} (t, \mathbf{x}(u, t_i^+)) \in [t_i, t_{i+1}] \times \mathbb{R}^2, \text{ such that} \\ \lim_{t \to t_i^-} \mathbf{x}(u,t) \neq \lim_{t \to t_i^+} \mathbf{x}(u,t) \\ \vee \\ \nexists \lim_{t \to t_i} \frac{\mathbf{x}(u,t) - \mathbf{x}(u,t_i)}{t - t_i} \end{array} \right\} \tag{20}$$

In this way, the treated vibro-impact system is finally expressed as a hybrid dynamic system, by joining the set C of continuous trajectories and the discrete events set D, satisfying the equality established in equation (4).

4.2 Inverse optimal control by passivity

The control strategy used to execute the designated task is shown in Fig. 3, where x_{1d} and x_{2d} are respectively the desired position and velocity, t_e corresponds to the instant of time of the event, and $\bar{x}_2$ is the velocity error variable of the tracking error vector established as

$$\bar{\mathbf{x}} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_{1d} \\ x_2 - x_{2d} \end{bmatrix}. \tag{21}$$

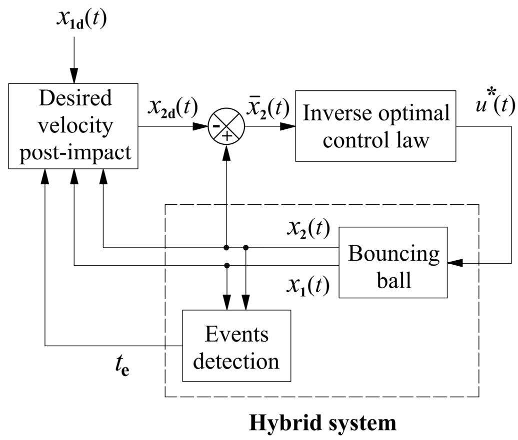

Fig. 3. General control scheme

It is necessary to clarify that due to the nature of the system, the position control is carried out through the ball velocity control after colliding with the rigid surface, and is executed for a short period of time post-impact. Therefore, according to (19), the controlled form of the continuous system given by (12) is established according to the equation (6) rewritten below

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})u(\mathbf{x}),$$

with

$$f(\mathbf{x}) = \begin{bmatrix} x_2 \\ -a_g \end{bmatrix}, \quad g(\mathbf{x}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

On the other hand, a Lyapunov function based on the velocity tracking error is assumed, so that this function has its global minimum along the desired trajectory.

Without loss of generality, based on (21), we modify the functional cost (7), the HJB equation (8), and the optimal control law (9), to achieve global asymptotic stability to the equilibrium point $\bar{\mathbf{x}} = 0$, such that

$$J = \int_0^\infty \left(l(\bar{\mathbf{x}}) + u^T(\bar{\mathbf{x}}) R(\bar{\mathbf{x}}) u(\bar{\mathbf{x}}) \right) dt, \tag{22}$$

$$l(\bar{\mathbf{x}}) + L_f V(\bar{\mathbf{x}}) - \frac{1}{4} L_g V(\bar{\mathbf{x}}) R^{-1}(\bar{\mathbf{x}}) \left(L_g V(\bar{\mathbf{x}}) \right)^T = 0, \tag{23}$$

$$u^*(\bar{\mathbf{x}}) = -\frac{1}{2} R^{-1}(\bar{\mathbf{x}}) \left(L_g V(\bar{\mathbf{x}}) \right)^T (\bar{\mathbf{x}}). \tag{24}$$

with $R(\bar{\mathbf{x}}) \in \mathbb{R}$, $R(\bar{\mathbf{x}}) > 0$ and $l(\bar{\mathbf{x}}) \in \mathbb{R}$, $l(\bar{\mathbf{x}}) \geq 0$. Also, the quadratic Lyapunov function proposed to minimize (23) is established as

$$V(\bar{\mathbf{x}}) = \frac{1}{2}\begin{bmatrix} 0 \\ \bar{x}_2 \end{bmatrix}^T K^T P K \begin{bmatrix} 0 \\ \bar{x}_2 \end{bmatrix}$$

where $P \in \mathbb{R}^{2\times 2}$ is a symmetric and definite positive matrix, and $K \in \mathbb{R}^{2\times 2}$ is an additional gain matrix introduced to modify the convergence rate of the tracking error. Finally, solving the lie derivative of equation (24), and assuming that

$$R(\bar{\mathbf{x}}) = 0.1, \quad P = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} \lambda & \lambda \\ \lambda & \lambda \end{bmatrix},$$

with $\lambda = 100$, the optimal stabilizing control law that satisfies (23) corresponds to

$$u^*(\bar{\mathbf{x}}) = -\frac{1}{2} R^{-1}(\bar{\mathbf{x}}) g^T(\mathbf{x}) P \begin{bmatrix} 0 & \bar{x}_2 \end{bmatrix}, \quad (25)$$

where T is the transpose operator $(\cdot)^T$.

5. RESULTS

In this section, numerical simulation validates the results obtained analytically. This process is carried out following the control scheme of the Fig. 3, using MATLAB® with the *ODE1* solver to a fixed-step of $50 \times 10^{-6}\ [s]$, and a simulation stop-time of $t_f = 5\ [s]$. Simulation parameters for the undisturbed case are specified in the Table 1.

Table 1: System parameters

Symbol	Description	Value
x_0	Initial position	2.5 $[m]$
v_0	Initial velocity	2 $[m/s]$
k	Restitution coefficient	0.8
x_{1d}	Maximum desired height	1 $[m]$
a_g	Gravity acceleration	9.8 $[m/s^2]$

In order to show the joint evolution of the system expressed in a hybrid way by means of equations (19) and (20), Fig. (4) shows the transition between continuous trajectories and discrete events during the moments when the impacts against the rigid surface. Additionally, in Fig. 5 it is shown how the maximum height requirement is reached, while in Fig. 6 the velocity curve is described, where is observed the fast and smooth way with which the ball acquires the necessary speed that satisfies the design requirement. On the other hand, Fig. 7 shows the efficiency with which the stability optimal control law manipulates the dynamics of the system to follow the desired trajectory, while confrontations of the state variables for the exposed case can be observed in the phase portrait of Fig. 8.

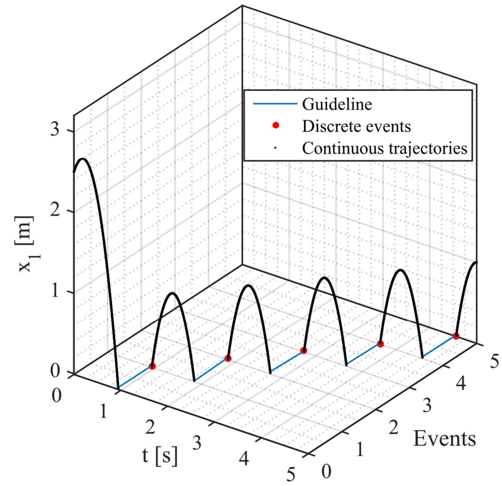

Fig. 4. Bouncing ball hybrid description

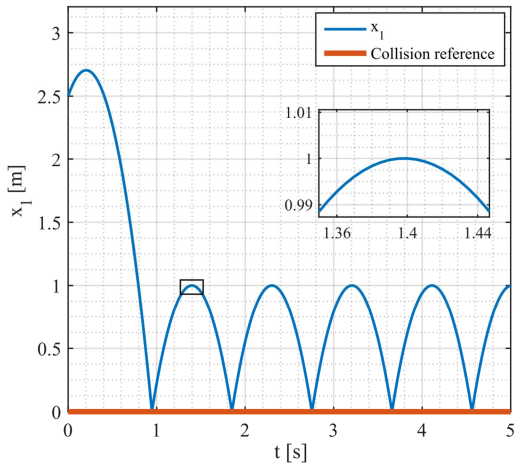

Fig. 5. Ball position

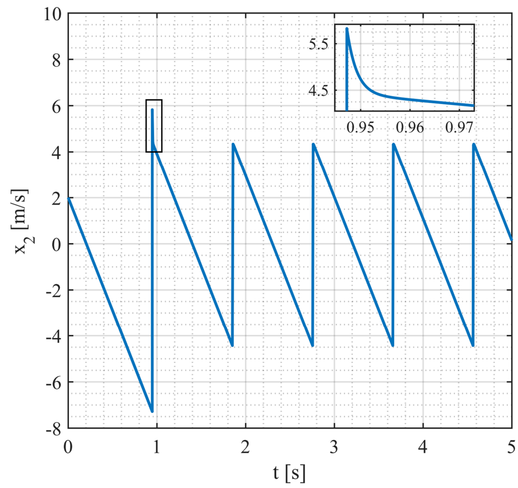

Fig. 6. Ball velocity

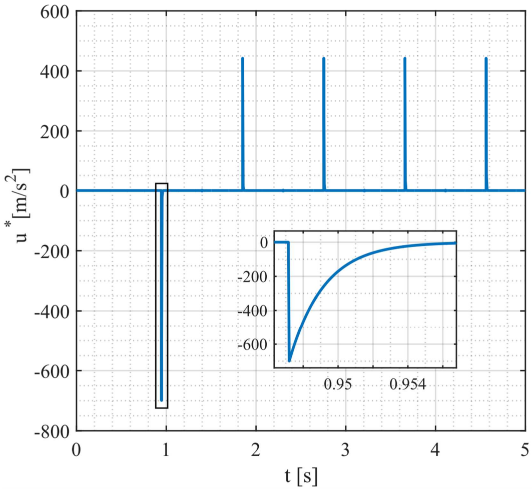

Fig. 7. Optimal law

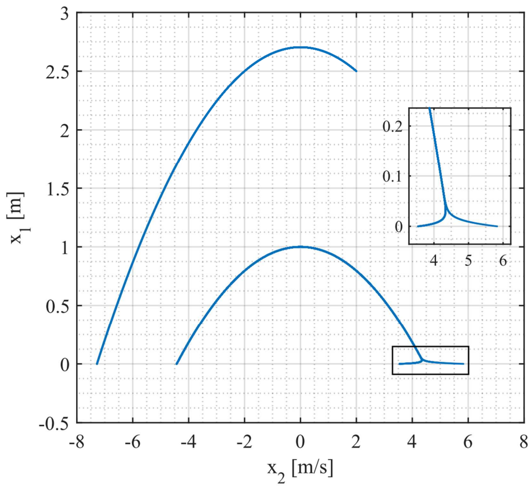

Fig. 8. Phase portrait

Additionally, it is considered the case in which the impact surface moves upwards with respect to the ground reference frame, behaving as a disturbance that gradually affects the flight time of the ball. Without loss of generality, the same simulation with parameters established in Table 1 are considered. So, in Fig. 9 the desired maximum height range is observed, even before the collision reference is raised. On the other hand, Fig. 10 it shows the smoothness with which the velocity desired is reached after impact, even when the collision reference changes in level. In the Fig. 11 the optimal control law for this case is displayed, where it is observed how its maximum amplitude decreases as the collision surface rises. Finally, by means of the phase map of Fig. 12 the way in which the system velocity executes an adequate desired trajectory tracking is viewed, despite the disturbance present at the reference level of the collision surface.

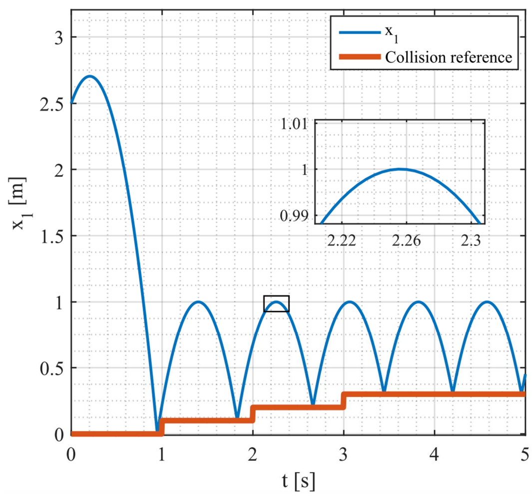

Fig. 9. Ball position – Disturbed

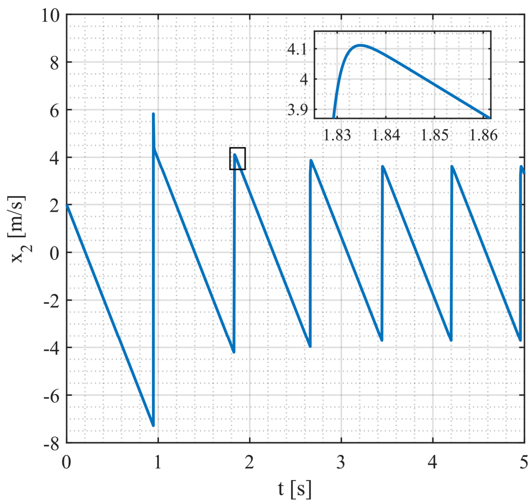

Fig. 10. Ball velocity - Disturbed

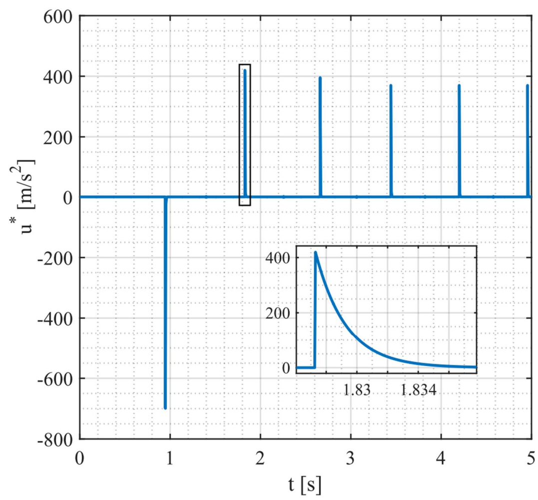

Fig. 11. Optimal law - Disturbed

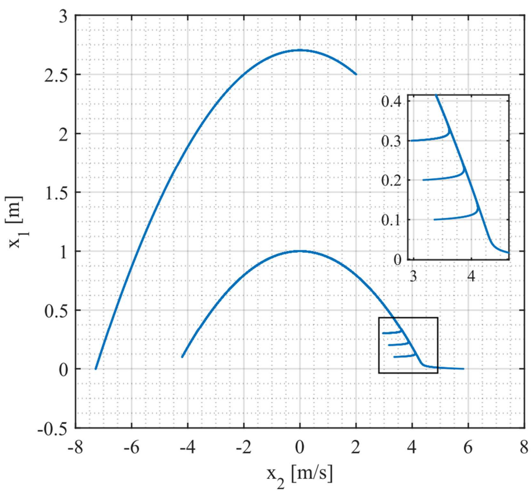

Fig. 12. Phase portrait - Disturbed

5. CONCLUSIONS

Based on the obtained results, the suitability of the proposed hybrid approach to represent systems with strong discontinuities in their dynamics is demonstrated, which considerably reduces the complexity of the model and allows the application of particular local control actions for each region of operation. On the other hand, the potential of the inverse approach to calculate an optimal stabilizing control law capable of forcing the system to follow the desired trajectory is evidenced; avoiding solving the HJB equation in exchange for satisfying a positive defined Lyapunov function with negative semi-definiteness, used as a storage function.

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